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ABSOLUTE MEASUREMENTS.

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IN

ELECTRICITY AND MAGNETISM.

BY

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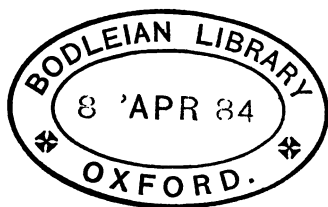
PREFACE.

THIS little book was originally intended to be mainly a reprint of some papers on the Measurement of Electric Currents and Potentials in Absolute Measure contributed to *Nature* during the winter of 1882-3; but as these were being reprinted, many alterations and additions suggested themselves, which it was thought would render the book more generally useful. Most of the additional matter is mentioned in the introductory chapter, but I may here refer to a sketch of the theory of alternating machines, and of methods of measurement available in such cases, contained in Chapter X., and to Chapter XII. on the Dimensions of Units, which I have thought it desirable to introduce.

The work has of course no pretensions to being a complete treatise on Electrical and Magnetic

Measurements, but is rather designed to give as far as is possible within moderate limits a clear account of the system of absolute units of measurement now adopted, and of some methods and instruments by which the system can be applied in both theoretical and practical work.

I am under great obligations to Sir William Thomson and to Mr. J. T. Bottomley, who have kindly examined some of the proofs, and favoured me with valuable suggestions.

<div style="text-align: right;">A. GRAY.</div>

THE UNIVERSITY, GLASGOW.
          October, 1883.

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ERRATA.

Page 77, line 10, for *C, Z, L,* read *c, z, l.*

  ,,   93, ,,   9, for "those" read "that."

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ABSOLUTE MEASUREMENTS

IN

ELECTRICITY AND MAGNETISM.

CHAPTER I.

INTRODUCTORY.

TWENTY years ago the experimental sciences of electricity and magnetism were in great measure mere collections of qualitative results, and, in a less degree, of results quantitatively estimated by means of units which were altogether arbitrary. These units, depending as they did on constants of instruments and conditions of experimenting which could never be made fully known to the scientific public, were a source of much perplexity and labour to every investigator, and to a great extent prevented the results which they expressed from bearing fruit to the furtherance of scientific progress. Now happily all this has been changed. The absolute system of units introduced by Gauss and Weber, and rendered a practical reality in this country by the labours of the British Association Committee on Electrical Standards, has changed experimental electricity and magnetism into sciences of which the very essence is the most delicate and exact measurement, and enables their results to be expressed in units which are

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altogether independent of the instruments, the surround-
ings, and the locality of the investigator.

The record of the determinations of units made by
members of the Committee, for the most part by methods
and instruments which they themselves invented, forms
one of the most interesting and instructive books[1] in the
literature of electricity, and when the history of electrical
discovery is written the story of their work will form one
of its most important chapters.   But besides placing
on a sure foundation the system of absolute units,
they conferred a hardly less important benefit on elec-
tricians by giving them a convenient nomenclature for
electrical quantities.   The great utility of the practical
units and nomenclature, which the Committee recom-
mended, soon became manifest to every one who had
to perform electrical measurements, and has led within
the last year to their adoption, with only slight al-
terations, by nearly all civilised nations.   Although it
is not yet quite two years since the late Congress
of Electricians at Paris concluded its sittings, the
recommendations which it issued have been widely
adopted and appreciated by those engaged in electrical
work, and have thus begun to yield excellent fruit by
rendering immediately available for comparison and as a
basis for further research the results of experimenters in
all parts of the world.   Soon even the ordinary workmen
in charge of dynamo machines or employed in electrical
laboratories will be able to tell the number of volts and
amperes which a generator can give at a certain speed
and under certain conditions, to determine the number of
amperes of current required to light an incandescence

[1] Reports of the British Association Committee on Electrical Standards
dited by Prof. Jenkin, F.R.S.

lamp to its full brilliancy, or to measure the capacity of a secondary cell in coulombs per square centimetre.

But in order that the full benefit of the conclusions of the Paris Congress may be obtained it is essential in the first place that convenient instruments should be used, adapted to give directly, or by an easy reduction from their indications, the number of amperes of current flowing in a particular circuit, and the number of volts of difference of potentials between any two points in that circuit. To be generally useful in practice these instruments should be easily portable, and should have a very large range of sensibility; so that, for example, the instrument, which suffices to measure the full potential produced by a large dynamo-electric machine, may be also available for testing, if need be, the resistances of the various parts of the armature and magnets by the only satisfactory method; namely by comparing by means of a galvanometer of high resistance the difference of potential between the two ends of the unknown resistance with that between the ends of a known resistance joined up in the same electrical circuit. In like manner the ampere measurer should be one that could be introduced without sensible disturbance into a circuit of low resistance to measure either a small fraction of an ampere, or the whole current flowing through a circuit containing a large number of electric lamps. These conditions are fulfilled by two instruments recently invented and patented by Sir William Thomson and called by him Graded Galvanometers. My main purpose in the present work is to give an account, both theoretical and practical, of the measurement of currents and potentials in absolute measure; and to apply this to the graduation of instruments for use in practical electrical work. Of such instruments

I shall take Sir William Thomson's Graded Galvano-meters as an example, and after describing them, show how they may be graduated, or their graduation tested by the experimenter himself. It will be convenient to introduce definitions of absolute units of measurement of magnetic and electrical quantities when they are required; and in so doing I shall endeavour to give a clear account of the foundation of the absolute system of electrical measurement, and to show how from the fundamental units are derived the practical units of current, quantity, potential, and resistance. I shall then give and explain a few rules for the calculation of currents and resistances in derived circuits; and describe among others some methods of determining resistances which are useful in many import-ant practical cases, but which so far as I know are not treated of in the ordinary text-books of electricity. The remainder of the work will contain a brief account of the measurement of energy spent in the circuits of generators transmitting power to electric lamps or motors, or in the charging circuit of a secondary battery; a chapter on the practical determination in absolute units of the intensities of powerful magnetic fields, such as those of the field magnets of dynamo machines; and, lastly, a few tables of useful electrical data.

CHAPTER II.

DETERMINATION OF THE HORIZONTAL COMPONENT OF THE EARTH'S MAGNETIC FIELD.

A LL the methods by which galvanometers may be graduated so as to measure currents and potentials in absolute units, involve, directly or indirectly, a comparison of the indications of the instrument to be graduated with those of a standard instrument, of which the constants are fully known for the place at which the comparison is made. There are various forms of such standard instruments, as, for example, the tangent galvanometer which Joule made, consisting of a single coil of large radius, and a small needle hung at its centre, or the Helmholtz modification of the same instrument with two large equal coils placed side by side at a distance apart equal to the radius of either; or some form of "dynamometer," or instrument in which the needle of the galvanometer is replaced by a movable coil, in which the whole or a known portion of the current in the fixed coil flows. The measurement consists essentially in determining the couple which must be exerted by the earth's magnetic force on the needle or suspended coil, in order to equilibrate that exerted by the current. But the former depends on the value, usually denoted by H, of the horizontal component of the earth's magnetic force, and it is

necessary therefore, except when some such method as that of Kohlrausch, described below, is used, to know the value of that quantity in absolute units.

The value of *H* may be determined in various ways, and I shall here content myself with describing the method which is most convenient in practice. It consists in finding (1) the angle through which the needle of a magnetometer is deflected by a magnet placed in a given position at a given distance, (2) the period of vibration of the magnet when suspended horizontally in the earth's field, so as to be free to turn round a vertical axis. The first operation gives an equation involving the ratio of the magnetic moment of the magnet to the horizontal component *H* of the terrestrial magnetic force, the second an equation involving the product of the same two quantities. I shall describe this method, which was given by Gauss, somewhat in detail.

A very convenient form of magnetometer is that devised by Mr. J. T. Bottomley, and made by hanging within a closed chamber, by a silk fibre from 6 to 10 cms. long, one of the little mirrors with attached magnets used in Thomson's reflecting galvanometers. The fibre is carefully attached to the back of the mirror, so that the magnets hang horizontally and the front of the mirror is vertical. The closed chamber for the fibre and mirror is very readily made by cutting a narrow groove to within a short distance of each end, along a piece of mahogany about 10 cms. long. This groove is widened at one end to a circular space a little greater in diameter than the diameter of the mirror. The piece of wood is then fixed with that end down in a horizontal base-piece of wood furnished with three levelling screws. The groove is thus placed vertical; and the fibre carrying the

mirror is suspended within it by passing the free end of
the fibre through a small hole at the upper end of the
groove, adjusting the length so that the mirror hangs
within the circular space at the bottom, and fixing the
fibre at the top with wax. When this has been done, the
chamber is closed by covering the face of the piece of
wood with a strip of glass, which may be either kept in
its place by cement, or by proper fastenings which hold it
tightly against the wood. By making the distance be-
tween the back and front of the circular space small, and
its diameter very little greater than that of the mirror,
the instrument can be made very nearly "dead beat,"
·that is to say, the needle when deflected through any
angle comes to rest at once, almost without oscillation
about its position of equilibrium. A magnetometer can
be thus constructed at a trifling cost, and it is much more
accurate and convenient than the magnetometers fur-
nished with long magnets frequently used for the deter-
mination of *H*; and as the poles of the needle may always
in practice be taken at the centre of the mirror, the calcu-
lations of results are much simplified.

The instrument is set up with its glass front in the
magnetic meridian, and levelled so that the mirror hangs
freely inside its chamber. The foot of one of the levelling
screws should rest in a small conical hollow cut in the
table or platform, of another in a V-groove the axis of
which is in line with the hollow, and the third on the
plane surface of the table or platform. When thus set up
the instrument is perfectly steady, and if disturbed can in
an instant be replaced in exactly the same position. A
beam of light passes through a slit, in which a thin ver-
tical cross-wire is fixed, from a lamp placed in front of the
magnetometer, and is reflected, as in Thomson's reflecting

galvanometer, from the mirror to a scale attached to the lamp-stand, and facing the mirror. The lamp and scale are moved nearer to or farther from the mirror, until the position at which the image of the cross-wire of the slit is most distinct is obtained. It is convenient to make the horizontal distance of the mirror from the scale for this position if possible one metre. The lamp-stand should also have three levelling screws, for which the arrangement of conical hollow, V-groove, and plane should be adopted. The scale should be straight, and placed with its length in the magnetic north and south line; and the lamp should be so placed that the incident and reflected rays of light are in an east and west vertical plane, and that the spot of light falls near the middle of the scale. To avoid errors due to variations of length in the scale, it should be glued to the wooden backing which carries it, not simply fastened with drawing pins as is often the case.

The magnetometer having been thus set up, four or five magnets, each about 10 cms. long and ·1 cm. thick, and tempered glass-hard, are made from steel wire. This is best done as follows. From ten to twenty pieces of steel wire, each perfectly straight and having its ends carefully filed so that they are at right angles to its length, are prepared. These are tied tightly into a bundle with a binding of iron wire and heated to redness in a bright fire. The bundle is then quickly removed from the fire, and plunged with its length vertical into cold water. The wires are thus tempered glass-hard without being seriously warped. They are then magnetized to saturation in a helix by a strong current of electricity. A horizontal east and west line passing through the mirror is now laid down on a convenient platform (made of wood

put together without iron and extending on both sides
of the magnetometer) by drawing a line through that
point at right angles to the direction in which a long thin
magnet hung by a single silk fibre there places itself.
One of these magnets is placed, as shown in Fig. 1, with
its length in that line, and at such a distance that a
convenient deflection of the needle is produced. This
deflection is noted and the deflecting magnet turned end
for end, and the deflection again noted. In the same way
a pair of observations are made with the magnet at the
same distance on the opposite side of the magnetometer;
and the mean of all the observations is taken. These
deflections from zero ought to be as nearly as may be
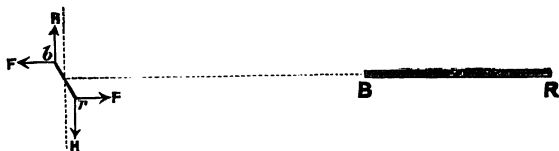
FIG. 1.

the same, and if the magnet is properly placed, they
will exactly agree; but the effect of a slight error in
placing the magnet will be nearly eliminated by taking
the mean deflection. The distance in cms. between the
two positions of the centre of the magnet is also noted
and is taken as twice the distance of the centre of the
magnet from that of the needle. The same opera-
tion is gone through for each of the magnets, which
are carefully kept apart from one another during the
experiments. The results of each of these experiments
give an equation involving the ratio of the magnetic
moment of the magnet to the value of H. Thus

if *m* denote the magnetic moment of the magnet, *m'* the magnetic moment of the needle, *r* the distance of the centre of the magnet from the centre of the needle, 2 *l* the distance between the poles [1] of the magnet, which, for a bar magnetized to saturation in a long helix, and of the dimensions stated above, is nearly enough equal to its length, and 2 *l'* the distance between the poles[1] of the needle, *r*, *l*, and *l'* being all measured in cms., we have for the repulsive force (denoted by F in Fig. 1) exerted on the blue [2] pole of the needle by the blue pole of the magnet, supposed nearest to the needle, as in Fig. 1, the value of $\frac{m}{2l} \cdot \frac{m'}{2l'} \cdot \frac{1}{(r-l)^2}$, since the value of *l'* is small compared with *l*. Similarly for the attraction exerted on the same pole of the needle by the red pole of the magnet, we have the expression $\frac{m}{2l} \cdot \frac{m'}{2l'} \cdot \frac{1}{(r+l)^2}$. Hence the total repulsive force exerted by the magnet on the blue pole of the needle is

$$\frac{m\,m'}{4\,l\,l'} \left\{ \frac{1}{(r-l)^2} - \frac{1}{(r+l)^2} \right\} \quad \text{or} \quad m\,\frac{m'}{l'} \cdot \frac{r}{(r^2-l^2)^2}.$$

Proceeding in a precisely similar manner, we find that the magnet *m* exerts an attractive force equal to $m\,\frac{m'}{l'} \cdot \frac{r}{(r^2-l^2)^2}$ on the red pole of the magnet. The needle is therefore acted on by a "couple" which tends to turn it round the suspending fibre as an axis, and the amount of this couple, when the angle of deflection is θ, is plainly equal to $m\,m'\,\frac{2\,r}{(r^2-l^2)^2}\cos\theta$. But for equilibrium this

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[1] See Note A. The needle may be considered as a very short uniformly magnetized magnet, the poles of which are therefore at its extremities.
[2] The convention according to which magnetic polarity of the same kind as that of the earth's northern regions is called blue, and magnetic polarity of the same kind as that of the earth's southern regions is called red, is here adopted. The letters B, R, *b*, *r*, in the diagrams denote blue and red.

couple must be balanced by $m'H\sin\theta$; hence we have the equation :—

$$\frac{m}{H} = \frac{(r^2 - l^2)^2}{2r}\cdot\tan\theta \quad . \quad . \quad . \quad . \quad (1)$$

If the arrangement of magnetometer and straight scale described above is adopted, the value of $\tan\theta$ is easily obtained, for the number of divisions of the scale which measures the deflection, divided by the number of such divisions in the distance of the scale from the mirror, is then equal to $\tan 2\theta$.

Instead of in the east and west horizontal line through the centre of the needle, the magnet may be placed, as represented in Fig. 2, with its length east and west, and its centre in the horizontal north and south line through the centre of the needle. If we take m, m', l, l', and r to have the same meaning as before, we have for the distance of either pole of the magnet from the needle, the expression $\sqrt{r^2 + l^2}$. Let us consider the force acting on one pole, say the red pole of the needle. The red pole of the magnet exerts on it a repulsive force, and the blue pole an attractive force. Each of these forces has the value $\frac{m}{2l}\cdot\frac{m'}{2l'}\cdot\frac{1}{r^2 + l^2}$. But the diagram shows that they are equivalent to a single force, F, in a line parallel to the magnet, tending to pull the red pole of the needle towards the left. The magnitude of this resultant force is plainly

$2\,\frac{m}{2l}\cdot\frac{m'}{2l'}\cdot\frac{l}{(r^2 + l^2)^{\frac{3}{2}}}$ or $\frac{m\,m'}{2\,l'\,(r^2 + l^2)^{\frac{3}{2}}}$. In the same way

it can be shown that the action of the magnet on the red pole of the needle is a force of the same amount tending to pull the blue pole of the needle towards the right. The needle is, therefore, subject to no force tending to

produce motion of translation, but simply to a "couple" tending to produce rotation. The magnitude of this couple when the needle has been turned through an angle

θ, is $\dfrac{m\,m'}{2\,l'} \cdot \dfrac{2\,l'\cos\theta}{(r^3 + l^2)^{\frac{3}{2}}}$, or $\dfrac{m\,m'}{(r^2 + l^2)^{\frac{3}{2}}}$ cos θ. If there be
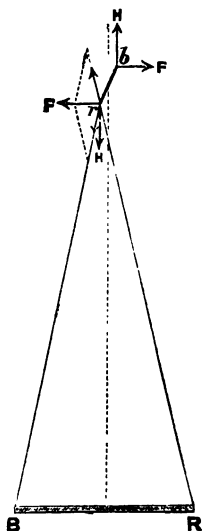
FIG. 2.

equilibrium for the deflection θ, this couple must be balanced by that due to the earth's horizontal force, which, as before, has the value $m'\,H\sin\theta$. Hence equating these two couples we have—

$$\frac{m}{H} = (r^2 + l^2)^{\frac{3}{2}} \tan\theta. \quad . \quad . \quad . \quad . \quad (2)$$

Still another position of the deflecting magnet relatively to the needle may be found a convenient one to adopt. The magnet may be placed still in the east and west line, but with its centre vertically above the centre of the needle. The couple in this case also is given by the formula just found, in which the symbols have the same meaning as before.

The greatest care should be taken in all these experiments, as well as in those which follow, to make sure that there is no movable iron in the vicinity, and the instruments and magnets should be kept at a distance from any iron nails or bolts there may be in the tables on which they are placed.

We come now to the second operation, the determination of the period of oscillation of the deflecting magnet when under the influence of the earth's horizontal force alone. The magnet is hung in a horizontal position in a double loop formed at the lower end of a single fibre of unspun silk, attached by its upper end to the roof of a closed chamber. A box about 30 cms. high and 15 cms. wide, having one pair of opposite sides, the bottom, and the roof made of wood, and the remaining two sides made of plates of glass, one of which can be slided out to give access to the inside of the chamber, answers very well. The fibre may be attached at the top to a horizontal axis which can be turned round from the outside so as to wind up or let down the fibre when necessary. The suspension-fibre is so placed that two vertical scratches, made along the glass sides of the box, are in the same plane with the magnet when the magnet is placed in its sling, and the box is turned round until the magnet is at right angles to the glass sides. A paper screen with a small hole in it is then set up at a little distance in such a position that the

hole is in line with the magnet, and therefore in the same plane as the scratches. The magnetometer should be removed from its stand and this box and suspended needle put in its place. If the magnet be now deflected from its position of equilibrium and then allowed to vibrate round a vertical axis, it will be seen through the small hole to pass and re-pass the nearer scratch, and an observer keeping his eye in the same plane as the scratches can easily tell without sensible error the instant when the magnet passes through the position of equilibrium. Or, a line may be drawn across the bottom of the box so as to join the two scratches, and the observer keeping his eye above the magnet and in the plane of the scratches may note the instant when the magnet, going in the proper direction, is just parallel to the horizontal line. The operator should deflect the magnet by bringing a small magnet near to it, taking care to keep this small deflecting magnet always as nearly as may be with its length in an east and west line passing through the centre of the suspended magnet. If this precaution be neglected the magnet may acquire a pendulum motion about the point of suspension, which will interfere with the vibratory motion in the horizontal plane. When the magnet has been properly deflected and left to itself, its range of motion should be allowed to diminish to about 3° on either side of the position of equilibrium before observation of its period is begun. When the amplitude has become sufficiently small, the person observing the magnet says sharply the word "Now," when the nearer pole of the magnet is seen to pass the plane of the scratches in either direction, and another observer notes the time on a watch having a seconds hand. With a good watch having a centre seconds

hand moving round a dial divided into quarter-seconds,
the instant of time can be determined with greater
accuracy in this way than by means of any of the usual
appliances for starting and stopping watches, or for regis-
tering on a dial the position of a seconds hand when a
spring is pressed by the observer. The person observing
the magnet again calls out " Now " when the magnet has
just made ten complete to and fro vibrations, again after
twenty complete vibrations, and, if the amplitude of vibra-
tion has not become too small, again after thirty ; and the
other observer at each instant notes the time by the watch.
By a complete vibration is here meant the motion of the
magnet from the instant when it passes through the position
of equilibrium in either direction, until it next passes through
the position of equilibrium going in the same direction.
The observers then change places and repeat the same
operations. In this way a very near approach to the true
period is obtained by taking the mean of the results of a
sufficient number of observations, and from this the value
of the product of *m* and *H* can be calculated.

For a small angular deflection θ of the vibrating
magnet from the position of the equilibrium the equation
of motion is

$$\frac{d^2\theta}{dt^2} + \frac{mH}{\mu}\theta = 0,$$

where μ is the moment of inertia of the vibrating magnet
round an axis through its centre at right angles to its
length. The solution of this equation is

$$\theta = A \sin\left\{ \sqrt{\frac{mH}{\mu}}\, t - B \right\}$$

and therefore for the period of oscillation *T* we have

$$T = 2\pi \sqrt{\frac{\mu}{mH}}$$

Hence we have

$$m\,H = \frac{4\,\pi^2\,\mu}{T^2}$$

Now, since the thickness of the magnet is small compared with its length, if W be the mass of the magnet μ is $\frac{l^2}{3}\,W$, and therefore

$$m\,H = \frac{4\,\pi^2\,l^2\,W}{3\,T^2} \quad . \quad . \quad . \quad . \quad (3)$$

Combining this with the equation (1) already found we get for the arrangement shown in Fig. 1,

$$m^2 = \frac{2}{3} \cdot \frac{\pi^2\,(r^2 - l^2)^3\,W \tan\theta}{T^2\,r} \quad . \quad . \quad (4)$$

$$H^2 = \frac{8}{3} \cdot \frac{\pi^2\,l^2\,r\,W}{T^2\,(r^2 - l^2)^2 \tan\theta} \quad . \quad . \quad (5)$$

If either of the other two arrangements be chosen we have from equations (2) and (3)

$$m^2 = \frac{4}{3} \cdot \frac{\pi^2\,l^2}{T^2}\,(r^2 + l^2)^{\frac{3}{2}}\,W \tan\theta \; . \quad . \quad (6)$$

$$H^2 = \frac{4}{3} \cdot \frac{\pi^2\,l^2\,W}{(r^2 + l^2)^{\frac{3}{2}}\,T^2 \tan\theta} \quad . \quad . \quad (7)$$

Various corrections which are not here made are of course necessary in a very exact determination of H. The magnetic moment of the deflecting magnet should be corrected for temperature. The approximate virtual length $2\,l$ of the magnet [1] may, if the length of the bar be sufficiently small in comparison with the distance of the centre from the needle, be found by observing the mean deflections θ and θ' of the magnetometer needle produced by the magnet when

[1] See Note A, on the Determination of H.

placed in the position shown in Fig 1. at distances r and r' from the centre of the needle. We have the equations

$$\frac{m}{H} = \frac{(r^2 - l^2)^2}{2\,r}\,tan\,\theta = \frac{(r'^2 - l^2)^2}{2\,r'}\,tan\,\theta'$$

and therefore,

$$l^2 = \frac{r'^2\,\sqrt{r\,tan\,\theta'} - r^2\,\sqrt{r'\,tan\,\theta}}{\sqrt{r\,tan\,\theta'} - \sqrt{r'\,tan\,\theta}} \quad . \quad . \quad . \quad (8)$$

Allowances should be made for the magnitude of the arc of vibration; the torsional rigidity of the suspension fibre of the magnetometer in the deflection experiments (p. 31), and of the suspension fibre of the magnet in the oscillation experiments; the frictional resistance of the air to the motion of the magnet ; the virtual increase of inertia of the magnet due to motion of the air in the chamber ; and the effect of induction in altering the moment of the magnet. The correction for an arc of oscillation of 6° is a diminution of the observed value of T of only $\frac{1}{60}$ per cent, and for an arc of 10° of $\frac{1}{20}$ per cent. Of the other corrections the last is no doubt the most important ; but its amount for a magnet of glass-hard steel, nearly saturated with magnetism, and in a field so feeble as that of the earth may, unless very great accuracy is aimed at, be neglected.

This correction arises from the fact that the magnet in the deflection experiments is placed in the magnetic east and west line, whereas in the oscillation experiments it is placed north and south, and is therefore subject in the latter case to an increase of longitudinal magnetisation from the action of terrestrial magnetic force. The increase of magnetic moment may be determined by the following method which is due to Mr. Thomas Gray. Place the magnet within, and near the centre of, a helix,

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considerably longer than the magnet and made of insulated copper wire. Place the helix and magnet in position either as shown in Fig. 2, or as in Fig. 3, for giving a deflection of the magnetometer needle, and read the deflection. Then pass such a current through the wire of the helix as will give by electro-magnetic induction a magnetic field within the helix nearly equal to the horizontal component of the earth's field, and again observe the deflection of the magnetometer needle. The intensity of the field thus produced within the helix at points not near the ends, is given in c.g.s. units (p. 24), by the formula $4\pi nC$, where n is the number of turns per centimetre of length of the helix, and C is the strength of the current in c.g.s. units (pp. 25—37). Experiments may be made with different strengths of current, and the results put down in a short curve, from which the correction can be at once read off when the approximate field has been determined by the method of deflection and oscillation described above. Care must of course be taken in experimenting to eliminate the deflection of the magnetometer needle caused by the current in the coil. This is easily done by observing the deflection produced by the current when the magnet is not inside the coil, and subtracting this from the previous deflection. The change of magnetic moment produced in hard steel bars, the length of which is 12 cms. and diameter ·2 cm., and previously magnetised to saturation, is, according to Mr. Thomas Gray's experiments, about $\frac{1}{10}$ per cent.

The deflection experiments are, as stated above, to be performed with several magnets, and when the period of oscillation of each of these has been determined, the magnetometer should be replaced on its stand, and the deflection experiments repeated, to make sure that the

magnets have not changed in strength in the meantime. The length of each magnet is then to be accurately determined in centimetres, and its weight in grammes; and from these data and the results of the experiments, the values of m and of H can be found for each magnet by the formulas investigated above. Equation (5) is to be used in the calculation of H when the arrangement of magnetometer and deflecting magnet, shown in Fig. 1, is adopted, equation (7), when that shown in Fig. 2 is adopted.

The object of performing the experiments with several magnets, is to eliminate as far as possible, errors in the determination of weight and length. The mean of the values of H, found for the several magnets, is to be taken as the value of H at the place of the magnetometer. We have now to apply this value to the measurement of currents.

CHAPTER III.

ABSOLUTE UNITS OF MAGNETIC POLE, MAGNETIC FIELD, AND ELECTRIC CURRENT.

I N the preceding investigation nothing has been said as to the units in which the quantities m and H are measured. It will be convenient, before proceeding further, to state how the absolute unit of current is defined in the absolute *electro-magnetic* system now generally adopted for most electrical measurements. This definition involves those of the absolute units of magnetic pole and magnetic field, which therefore must be considered first.

In the *electro-magnetic system* of measurement, all magnetic and electrical quantities are expressed in units which are derived from a magnetic pole chosen as the pole of unit strength. This unit pole might be defined in many ways; but in order to avoid the fluctuations to which most arbitrary standards would be subject, and to give a convenient system in which work done in the displacements of magnets or conductors, relatively to magnets or to conductors carrying currents, may be estimated without the introduction of arbitrary and inconvenient numerical factors, it is connected by definition with the absolute unit of force. It is defined as *a pole which, if placed at unit distance from an equal and similar pole would be repelled with unit force*.[1] The poles referred to

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[1] The medium between the poles is supposed to be air.

in this definition are purely ideal, for we cannot separate one pole of a magnet from the opposite pole of the same magnet : but we can by proper arrangements obtain an approximate realisation of the definition. Suppose we have two long, very thin, straight, steel bars, which are uniformly and longitudinally magnetised ; their poles may be taken as at their extremities ; in fact, the distribution of magnetism in them is such that the magnetic effect of either bar, at all points external to its own substance, would be perfectly represented by a certain quantity of one kind of imaginary magnetic matter placed at one extremity of the bar, and an equal quantity of the opposite kind of matter placed at the other extremity. We may imagine, then, these two bars placed with their lengths in one line, and their blue poles turned towards one another, and at unit distance apart. If their lengths be very great compared with this unit distance, say 100 or 1000 times as great, their red poles will have no effect on the blue poles comparable with the repulsive action of these on one another. But there will be an inductive action between the two blue poles which will tend to diminish their mutual repulsive force, and this we cannot in practice get rid of. The magnitude of this inductive effect is, however, less for hard steel than for soft steel, and we may therefore imagine the steel of the magnets so hard that the action of one on the other does not appreciably affect the distribution of magnetism in either. If, then, two equal blue poles repel one another with unit force, each according to the definition has unit strength.

The magnitude of unit pole is by the above definition made to depend on unit force. Now unit force is defined, according to the system of measurement of forces founded on Newton's Second Law of Motion, the most convenient

system, as that force which, acting for unit of time on
unit of mass, will give to that mass unit of velocity.
The unit pole is thus based on the three fundamental
units of length, mass, and time.   According to the recom-
mendations of the B.A. Committee, and the resolutions of
the Paris Congress, it has been resolved to adopt generally
the three units already in very extended use for the ex-
pression of dynamical, electrical, and magnetic quantities,
namely, the centimetre as unit of length, the gramme as
unit of mass, and the second as unit of time ; and these
units are designated by the letters *c.g.s.*   With these
units, therefore, unit force is that force which, acting for
one second on a gramme of matter, generates a velocity of
one centimetre per second.   This unit of force has been
called a *dyne.*   The unit magnetic pole, therefore, in the
c.g.s. system of units is that pole which, placed at a
distance of 1 centimetre from an equal and similar pole, is
repelled with a force of 1 dyne.   Each of the poles of the
long thin magnets of our example above is therefore a
pole of strength equal to one c.g.s. unit, if the mutual
force between the poles is 1 dyne.

The magnetic moment *m* of any one of the deflecting
magnets is equal to the strength of either pole multiplied
into the distance between the poles, which for magnets of
such great length in comparison with their thickness is
nearly enough the actual length of the magnet.   There-
fore either pole has a strength of $\frac{m}{2l}$ units.   If *r* and *l* are
measured in centimetres, and *W* in grammes, the strengths
of the magnetic poles deduced from equation (4) or (6)
will be in c.g.s. units.

A magnetic field is the space surrounding a magnet or
a system of magnets, or a system of conductors carrying

currents, at any point of which, if a magnetic pole were placed, it would be acted on by magnetic force. The intensity of the magnetic field due to any system of magnets or of conductors carrying currents, or any combination of such systems, is measured at every point by the force which a magnetic pole would there experience, and the direction and magnitude of this force can, theoretically, be calculated if the magnetic distribution is given. Hence from the definition of unit magnetic pole we get at once a definition of magnetic field of unit intensity. *Unit magnetic field is that field in which unit magnetic pole is acted on by unit force*, and in the c.g.s. system, therefore, it is that field in which unit magnetic pole is acted on by a force of one dyne. In the theory of the determination of H, given above, the horizontal force on either pole of the magneto-meter needle due to the horizontal component of the earth's field is taken as $\dfrac{m'}{2\,l'}$. H, and again the horizontal force on either pole of the deflecting magnet as $\dfrac{m}{2\,l}$. H. H therefore measures in units of magnetic field intensity the horizontal component of the earth's field. By formula (5) or (7), when r and l are taken in centimetres, and W in grammes, H is given in dynes; that is, it is the number of dynes with which a unit red pole would be pulled horizontally towards the north, and a unit blue pole towards the south if acted on only by the earth's magnetic field. In the magnetic field due to a single magnetic pole, the direction of the resultant magnetic force at any point is in the straight line joining the point with the pole producing the field, and the magnitude of this force, which measures the intensity of the field at the point, is the force which a unit pole would there experience, and is measured

by the strength of the pole producing the field, divided by the square of its distance from the point in question.

According to the theory of electro-magnetic action given by Ampère, a plane closed circuit carrying a current in a magnetic field, and of very small dimensions in comparison with its distance from any part of the magnetic system producing the field, is equivalent, as regards magnetic action, to a small magnet having a magnetic moment directly proportional to the strength of the current, placed with its length at right angles to the plane of the circuit at some point within it, and so turned that to an observer towards whom the red pole is pointing, the current circulates in the direction opposite to the motion of the hands of a watch. If we measure the current in such units that the magnetic moment of the equivalent magnet is equal numerically to the strength of the current multiplied by the area of the circuit, we define unit current (1) as *that current which flowing in a small plane circuit, is equivalent*[1] *in magnetic action to a small magnet of moment numerically equal to the area of the circuit.*

From this result for a small plane closed circuit it follows that the mutual action between a finite closed circuit of any form carrying a current in a magnetic field and the magnetic system producing the field is the same as that which would exist if the circuit were replaced by any magnetic shell (that is a thin material surface magnetised so as to have, on one side, blue magnetism, and on the other side red magnetism,) whose bounding edge occupies the position of the circuit, provided the shell does not pass

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[1] A circuit and magnet equivalent in one medium are not necessarily equivalent in another medium. It is assumed here as elsewhere that the medium is air.

through any part of the magnetic system, and is magnet-
ised in the proper direction, and so that the magnetic
moment of every small part is equal to the strength of the
current multiplied into the area of that part.

By considering the work (p. 45) done in any small dis-
placement of an element of the boundary of an equivalent
shell, it can be shown that every element of the circuit
is acted on directly by a force, at right angles to the
plane through the element and the direction of the
resultant magnetic force at its centre, the magnitude of
which is obtained by multiplying together the length of
the projection of the element on a plane at right angles to
the direction of the resultant magnetic force, the magni-
tude of this force, and the strength of the current. Thus
if ds denote the length of the element, I the resultant
magnetic force, or, which is the same, the intensity of the
field at the element, θ the angle between the length of the
element and the direction of the magnetic force, C the
strength of the current, and dF the force on the element,
we have $dF = C I d s \sin \theta$. If we take $\theta = 90°$, and
$I = 1$, we get $dF = C d s$, and hence the definition :—(2)
*Unit current is the current flowing in an element of a
conductor which placed at right angles to the direction of
the resultant force in a field of unit intensity at the element,
is acted on by an electro-magnetic force, the amount of
which per unit of length of the element is equal to unity.*
If an observer be supposed immersed in the current so
that it flows from his feet to his head, and to look along
the line of the resultant magnetic force in the direction in
which it would move a blue magnetic pole, the element if
free to move would do so towards his right hand.

If we suppose the magnetic system to consist of a
single magnetic pole, and put r for the distance of the

pole from the element, and μ for the strength of the pole, I becomes $\frac{\mu}{r^2}$, and we have $dF = \frac{C\mu}{r^2}\,ds\,\sin\theta$.

Considering now the opposite aspect of the stress in this case, the action of each element of the circuit on the pole is, by Newton's law, that action and reaction are equal and opposite, a force equal to dF reversed, and therefore acting *in the same line*. But this is equivalent to an equal and similar parallel force acting in a line through the pole; together with a couple, the moment of which is $r\,dF$, acting in the plane of r and dF. For a closed circuit the resultant moment of the couples for all the elements is zero, and the total action is the same as if the forces alone acted on the pole.

From these considerations we may define unit current as (3) *that current which, flowing in a thin wire forming a circle of unit radius, acts on a unit magnetic pole placed at the centre with unit force per unit of length of the circumference*, that is, with a total force of 2π units ; or which produces at the centre of the circle a magnetic field of 2π units of intensity. Thus in the c.g.s. system unit current is that current which flowing in a circle of unit radius acts on a unit magnetic pole at the centre with a force of 2π dynes.

This force acts towards one side or the other of the plane of the circle, according to the nature of the pole and the direction of the current, and its direction may be found for any case by remembering that the earth may be imagined to be a magnet turned into position by the action of a current flowing round the magnetic equator in the direction of the sun's apparent motion.

These three definitions of unit current are equivalent, and in the applications which follow we shall use that which is most convenient in any particular case.

CHAPTER IV.

MEASUREMENT OF A CURRENT IN ABSOLUTE UNITS AND PRACTICAL CONSTRUCTION OF A STANDARD GALVANOMETER.

IF we take next the simple case of a single wire bent round into a circle and fixed in the magnetic meridian, with a magnet, whose dimensions are very small in comparison with the radius of the wire, hung by a torsionless fibre so as to rest horizontally with its centre at the centre of the circle, we may suppose that each pole of the magnet is at the same distance from all the elements of the wire. A current flowing in the wire acts, by Ampère's theory, with a force on one pole of the needle towards one side of the plane of the circle, and on the other pole with an equal force towards the other side of that plane. The needle is thus acted on by a couple tending to turn it round, and it is deflected from its position of equilibrium until this couple is balanced by the return couple due to H. Let us suppose the strength of each pole of the needle to be m units, r the radius of the circle, and C the strength of the current in it. Then by Ampère's law we have for the whole force without regard to sign, exerted on either pole of the needle by the current, the value $Cm\,\dfrac{2\pi r}{r^2}$ or $Cm\,\dfrac{2\pi}{r}$. If l be the length of

the needle the couple is $C m \dfrac{2\pi}{r} l$, before any deflection has taken place. After the needle has been deflected through the angle θ the arm l of the couple has become $l \cos \theta$, and therefore the couple $C m \dfrac{2\pi}{r} l \cos \theta$; and the return couple due to H is $m H l \sin \theta$. Hence we have equilibrium when

$$C m \frac{2\pi}{r} l \cos \theta = m H l \sin \theta$$

and therefore

$$C = \frac{H r}{2 \pi} \tan \theta \quad . \quad . \quad . \quad . \quad (1)$$

if θ be the observed angle at which the needle rests in equilibrium when deflected as described from the magnetic meridian. If instead of a single circular turn of wire we had N turns occupying an annular space of mean radius r, and of dimensions of cross-section small compared with r we should have

$$C = \frac{H r}{2 \pi N} \tan \theta \quad . \quad . \quad . \quad . \quad (2)$$

In practice the turns of wire of the tangent galvanometer may not be all contained within such an annular space. It is necessary then to allow for the dimensions of the space occupied by the wire. For a coil made of wire of small section we may suppose that the actual current flowing across a unit of area is everywhere the same. Hence if C be the current strength in each turn, and n the number of turns in unit area, we have for the current crossing the area A of an element E the value $n C A$. Taking a section of the coil through its centre, at right angles to its plane, and supposing the

annular space of rectangular section, let $B\,C$ (Fig. 3) be a
radius drawn from the centre C in the plane cutting the coil
into two equal and similar coils, and taking $C\,D\,(=y)$
and $D\,E\,(=x)$ at right angles to one another, we have
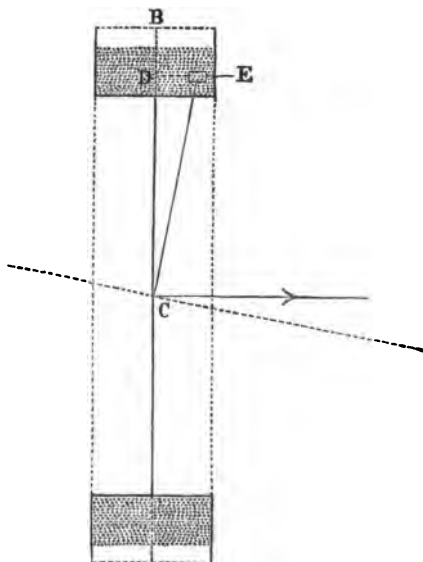
Fig. 3.

$A = d\,x\,d\,y$ and $C\,E^2 = x^2 + y^2$. Hence the force exerted
on a unit magnetic pole at the centre C by the ring
supposed at right angles to the plane of the paper, of which
this element is the section, will be $\dfrac{2\,\pi\,n\,C\,y\,d\,x\,dy}{x^2 + y^2}$ in the

direction at right angles to CE and in the plane of the paper. If we call the component of this force at right angles to BC, dF, we have

$$dF = \frac{2\pi n\, C y^2\, dx\, dy}{(x^2 + y^2)^{\frac{3}{2}}}.$$

Hence for the whole force at right angles to BC we have

$$F = 2\pi n\, C \int_{-b}^{b} \int_{r-c}^{r+c} \frac{y^2\, dx\, dy}{(x^2 + y^2)^{\frac{3}{2}}},$$

where r is the mean radius of the coil, $2b$ its breadth, and $2c$ its depth in the plane of the circle.

Integrating, and putting N for the whole number of turns $4\,nbc$, we get

$$F = \pi N C \frac{1}{c} \log \frac{r + c + \sqrt{(r+c)^2 + b^2}}{r - c + \sqrt{(r-c)^2 + b^2}} . \quad (3)$$

if $\dot{\theta}$ be the angle at which the deflecting couple is equilibrated by the return couple due to H, we have as before the equation

$$F = H \tan \theta.$$

Hence, substituting the above value for F and solving for C, we have finally

$$C = \frac{H \tan \theta}{\pi N \dfrac{1}{c} \log \dfrac{r + c + \sqrt{(r+c)^2 + b^2}}{r - c + \sqrt{(r-c)^2 + b^2}}} . \quad (4)$$

When the value of r is great in comparison with b and c this reduces to the equation

$$C = \frac{H r \tan \theta}{2\pi N} \quad . \quad . \quad . \quad . \quad (5)$$

which we found before by assuming all the turns to be contained in a small annular space of radius r. In practice, in galvanometers used as standards for absolute

measurements, generally neither b nor c is so great as $\frac{1}{10}$ of r, and the needle cannot be made infinitely short ; hence in these cases the difference between the values given by equations (4) and (5) is well within the limits of experimental error, and the correction need not be made. The value of C given by (5) is then to be used.

When the dimensions of the coil are measured in centimetres, and H is taken in c.g.s. units the value of C is given by (4) or (5) in c.g.s. units of current strength. In this investigation the suspension fibre has been supposed torsionless. If a single fibre of unspun silk is used as described below for this purpose, its torsion may for most practical purposes be safely neglected. The error produced by it may however be easily determined and allowed for by turning the needle, supposed initially in the magnetic meridian, once or more times completely round, and noting its deviation from the magnetic meridian in its new position of equilibrium. The amount of this deviation, if any, may be easily observed by means of the attached index and divided circle, or reflected beam of light and scale, used as described below, to measure the deflections of the needle. From the result of this experiment the effect of torsion for any deflection may be calculated in the following manner.

Let a be the angular deflection, in radian [1] measure, of the magnet from the magnetic meridian produced by turning the magnet once round, then the angle through which the thread has been twisted is $2\pi - a$. The couple produced by this torsion has for moment $H l m \sin a$. Hence, by Coulomb's law of the proportionality of the

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[1] A *radian* is the angle subtended at the centre of a circle by an arc equal in length to the radius. It has generally been called in books on trigonometry hitherto by the ambiguous name *unit angle in circular measure*.

force of torsion to the twist given, we have for the couple corresponding to a deflection θ the value

$$\frac{\theta}{2\pi - a} H m l \sin a.$$

If then under the action of a current in the coil the deflection of the needle is θ, the equation of equilibrium is

$$C m \frac{2\pi}{r} l \cos \theta = m H l \left(\sin \theta + \frac{\theta}{2\pi - a} \sin a \right)$$

and therefore instead of (5) we have

$$C = \left(1 + \frac{\theta}{2\pi - a} \frac{\sin a}{\sin \theta} \right) \frac{H r}{2\pi N} \tan \theta. \quad . \ (6)$$

If a be an angle of say 1°, and θ be 45°, $\dfrac{\theta}{2\pi - a}$ is very

nearly $\dfrac{1}{8}$ and $\dfrac{\sin a}{\sin \theta}$ is $\dfrac{1}{57\cdot3} \times \dfrac{1}{\cdot707}$ or $\dfrac{1}{40\cdot5}$.   Hence

$$C = \left(1 + \frac{1}{324} \right) \frac{H r}{2\pi N}.$$

The error therefore is somewhat less than $\frac{1}{3}$ per cent.

The determination of H and the measurement of a current in absolute units, can be effected simultaneously by the method devised by Kohlrausch, and described in the *Philosophical Magazine*, vol. xxxix. 1870. This method consists essentially in sending the current to be measured through two coils, of which all the constants are accurately known. One of these is the coil of a standard galvanometer, the other is a coil hung by a bifilar suspension, the wires of which convey the current into the coil. The latter coil rests in equilibrium when no current is passing through it, with its plane in the magnetic meridian. When a current is sent through it, it is acted on by a couple due to electro-magnetic action

between the current and the horizontal component of the earth's force, which tends to set it with its plane at right angles to the magnetic meridian; and this couple is resisted by the action of the bifilar. The coil comes to rest, making a certain angle with the magnetic meridian, and as the couple exerted by the bifilar suspension for any angle is supposed to have been determined by experiment, a relation between the value of H and the value of the current is obtained. But, as the same current is sent through the coil of the standard galvanometer, the observed deflection of the needle of that instrument gives another relation between H and C. From the two equations expressing these relations the values of H and C can be found. Full details of the construction of Kohlrausch's apparatus and of the calculation of its constants will be found in the paper above referred to.

In this method it is assumed that the value of H is the same at both instruments, an assumption which for rooms not specially constructed for magnetic experiments cannot safely be made. An instrument which is not liable to this objection has been suggested by Sir William Thomson. A short account of this instrument and its theory will be found in Maxwell's *Electricity and Magnetism*, vol. ii. p. 328.

The accuracy of the measurements of currents, made according to the method of which the theory is given above, of course altogether depends on the careful adjustment of the standard galvanometer, and the care and skill of the observer. The standard galvanometer should be of such a form that the values of its indications can be easily calculated from the dimensions and number of turns of wire in the coil. Such a galvanometer can be made by any one who can turn or can get turned a wooden

or, preferably, brass ring with a rectangular groove round
its outer edge to receive the wire.   It is indeed to be pre-
ferred that the experimenter should at least perform the
winding of the coil and the adjustments of the needle, &c.,
himself, to make sure that errors in counting the number
of turns or in determining the length of the wire, or in
placing the needle at the centre of the coil, are not made.
The breadth and depth of this groove ought to be small
in comparison with its radius, and each should not be
greater than $\frac{1}{10}$ of the mean radius of the coil, which should
be at least 15 cms.   The size of the wire with which
the coil is to be wound must depend of course on the
purposes to which the instrument is to be applied, but
it should be good well insulated copper wire of high
conductivity, and not so thin as to run any risk of being
injured by the strongest currents likely to be sent through
the instrument.   For the exact graduation of current as
well as potential galvanometers directly by means of the
standard instrument, it is convenient to make it have
two coils—one of comparatively high, the other of low
resistance.   The latter may conveniently in some cases
be a simple hoop of say 15 cms. radius, made of copper
strip 1 cm. broad and 1 mm. thick.   To form electrodes
to which wires can be attached the ends of the strip are
brought out side by side in the plane of the ring with
a piece of thin vulcanite or paper between for insulator.
Insulated wires are soldered to the ends of the circle
thus arranged, and are twisted together for a sufficient
distance to prevent any direct effect on the needle from
being produced by a current flowing in them.   In con-
structing the other coil the operator should first subject
the wire to a moderate stretching force, and then care-
fully measure its electrical resistance and its length.   He

should then wind it on a moderately large bobbin, and again measure its resistance. If the second measurement differs materially from the first, the wire is faulty and should be carefully examined. If no evident fault can be found on the removal of which the discrepance disappears, the wire must be laid aside and another substituted. When the two measurements are found to agree the wire may then be wound on the coil. For this purpose the ring may either be turned slowly round in a lathe or on a spindle, so as to draw off the wire from the bobbin, also mounted so as to be free to turn round. The wire must be laid on evenly in layers in the groove, and the winding ended with the completion of a layer. Great care must be taken to count accurately the number of turns laid on. The resistance should now be again tested, and if it agrees nearly with the former measurements the coil may be relied on. The ring carrying the coil thus made should then be fixed to a convenient stand in such a manner that if necessary it can be easily removed. The stand ought to be fitted with levelling screws, so that the plane of the coil may be made accurately vertical. A shallow horizontal box with a glass cover and mirror bottom should be carried by the stand at the level of its centre. Within this the needle and attached index are to be suspended. The needle should be a single small magnet about a centimetre long, hung by a single fibre of unspun silk about 10 cms. long from the top of a tube fixed to the cover of the shallow box, so that the centre of the needle when the coil is vertical is exactly at the centre of the coil. To allow of the exact adjustment of the height of the needle, the fibre should be attached to the lower end of a small screw spindle, made so as to be raised or lowered, without being turned round, by a nut working

round it above the cap of the tube. The needle should carry a thin glass index, about 6 inches long, made by drawing out a bit of thin glass tube at the blowpipe. In order that the zero position of the index may not be under the coil, the index ought to be fixed horizontally with its length at right angles to the needle, so as to project to an equal distance in both sides of it. To test that this adjustment is accurately made, draw a couple of lines accurately at right angles to one another on a sheet of paper. Then suspend a long thin straight magnet over the paper, and bring one of the lines into accurate parallelism with it. Remove then the magnet and put in its place the little needle and attached index. If the index is parallel to the other line the adjustment has been carefully made. The needle may then be suspended in position and the box within which it hangs closed to prevent disturbance from currents of air.

, A circular scale graduated to degrees, with its centre just below the centre of the coil and its plane horizontal, is placed with its zero point on a line drawn on the mirror bottom of the box at right angles to the plane of the coil, so that when the needle and coil are in the magnetic meridian the index may point to zero. The accuracy of the adjustment of the zero point is to be tested by finding whether the same current reversed produces equal deflections on the two sides of zero. To test whether the centre of this divided circle is accurately under the centre of the needle supposed at the centre of the coil, draw from the point immediately under the centre of the needle two radial lines on the mirror bottom, one on each side of the zero point and 45° from it, and turn the needle round without giving it any motion of translation. If the index lies along these two radial lines when its point is at the

corresponding division on the circle the adjustment is correct.

When taking readings the observer places his eye so as to see the index just cover its image in the mirror bottom of the box, and reads off the number of divisions and fraction of a division, indicated on the scale by the position of the index.   Error from parallax is thus avoided.

A mirror with attached magnets may be used, as in the magnetometer, instead of the needle and index.   When this arrangement is employed the coil is in the magnetic meridian when equal deflections of the spot of light on the scale on the two sides of zero are produced by reversing any current.   These scales, as has been already remarked, should always be carefully glued to a wooden piece, instead of being, as they frequently are, fixed with drawing pins.

CHAPTER V.

DEFINITION OF ABSOLUTE UNITS OF POTENTIAL AND
RESISTANCE, AND DERIVATION OF PRACTICAL
UNITS—VOLT, OHM, AMPERE, COULOMB. RATE
OF WORKING IN AN ELECTRIC CIRCUIT.

TWO conductors of the same material, are at different
potentials if on their being put in conducting con-
tact electricity tends to pass from one to the other. A
difference of potentials may also exist between two con-
ductors of different materials in contact, or may be
maintained between two parts of the same conductor by
electrical forces. In all cases a difference of potentials is
measured by the work (p. 45) which would be done by
electrical forces on unit quantity of positive electricity if,
while the potentials remained the same, it were carried
from the place of higher to the place of lower potential.
Two bodies at different potentials attract one another ;
and therefore, if one be connected with an electrically
insulated plate carried at one end of a delicate balance,
and the other with a second insulated plate fixed facing
the first at a very short distance from it, these two plates
will, if they have been previously put for an instant in
conducting contact and balance obtained, attract one
another, and the force of attraction may be weighed by

restoring balance.   With certain arrangements necessary
to ensure accuracy, a balance may be constructed by
means of which the difference of potentials between two
conductors can be measured.   Such an instrument has
been made by Sir William Thomson, and called by him
an Absolute Electrometer.

It is found experimentally by measuring with a delicate
electrometer that, between any two cross-sections A and
B of a homogeneous wire, in which a uniform current of
electricity is kept flowing by any means, there exists a
difference of potentials, and that if the wire be of uniform
section throughout, the difference of potentials is in direct
proportion to the length of wire between the cross-sections.
It is found further, that if the difference of potentials
between A and B is kept constant, and the length of wire
between them is altered, the strength of the current varies
inversely as the length of the wire.   The strength of the
current is thus diminished when the length of the wire is
increased, and hence the wire is said to oppose *resistance*
to the current ; and the resistance between any two cross-
sections is proportional to the length of wire connecting
them.   If the length of wire and the difference of poten-
tials between A and B be kept the same, while the cross-
sectional area of the wire is increased or diminished, the
current is increased or diminished in the same ratio ; and
therefore the resistance of a wire is said to be inversely
as its cross-sectional area.   Again, if for any particular
wire, measurements of the current strength in it be made
for various measured differences of potentials between
its two ends, the current strengths are found to depend
only on, and to be in simple proportion to, the differences
of potential so long as there is no sensible heating of the
wire.   Hence we have Ohm's law, by which to find the

strength C of the current flowing in a wire of resistance R, between the two ends of which a difference of potentials V is maintained, namely

$$C = \frac{V}{R}.$$

In this equation the units in which any one of the three quantities is expressed depend on those chosen for the other two. We have defined unit current, and have seen how to measure currents in absolute units ; and we have now to show how the absolute units of V and R are to be defined, and from them and the absolute unit of current to derive the practical units—volt, ampere, coulomb, and ohm.

We shall define the absolute units of potential and resistance by a reference to the action of a very simple but ideal magneto-electric machine, of which, however, the modern dynamo is merely a practical realisation. First of all let us imagine a uniform magnetic field of unit intensity. The lines of force in that field are everywhere parallel to one another : to fix the ideas let them be vertical. Now imagine two straight horizontal metallic rails running parallel to one another, and connected together by a sliding bar, which can be carried along with its two ends in contact with them. Also let the rails be connected by means of a wire so that a complete conducting circuit is formed. Suppose the rails, slider, and wire to be all made of the same material, and the length and cross-sectional area of the wire to be such that its resistance is very great in comparison with that of the rest of the circuit, so that, when the slider is moved with any given velocity, the resistance in the circuit remains practically constant. When the slider is moved along the rails it cuts across the lines of force, and so long as it moves with uniform velocity a constant difference of potentials

will be maintained between its two ends by induction, and
a uniform current will flow in the wire from the rail which
is at the higher potential to that which is at the lower. If
the direction of the lines of force be the same as the
direction of the vertical component of the earth's magnetic
force in the northern hemisphere, so that a blue pole
placed in the field would be moved upwards, and if the rails
run south and north, the current when the slider is moved
northwards will flow from the east rail to the west through
the slider, and from the west rail to the east through the
wire.   If the velocity of the slider be increased the differ-
ence of potentials between the rails, or, as it is otherwise
called, the electromotive force producing the current, will
be increased in the same ratio; and therefore by Ohm's law
so also will the current.   Generally for a slider arranged as
we have imagined, and made to move across the lines of
force of a magnetic field, the difference of potentials pro-
duced would be directly as the field intensity, as the
length of the slider, and as the velocity with which the
slider cuts across the lines of force.   The difference of
potentials produced therefore varies as the product of
these three quantities; and when each of these is unity,
the difference of potentials is taken as unity also.   We
may write therefore $V = I L v$, where I is the field in-
tensity, L the length of the slider, and v its velocity.
Hence if the intensity of the field we have imagined be
1 c.g.s. unit, the distance between the rails 1 cm., and the
velocity of the slider 1 cm. per second, the difference of
potentials produced will be 1 c.g.s. unit.

   This difference of potentials is so small as to be incon-
venient for use as a practical unit, and instead of it the
difference of potentials which would be produced if, every-
thing else remaining the same, the slider had a velocity

of 100,000,000 cms. per second, is taken as the practical
unit of electromotive force, and is called one *volt.* It is
a little less than the difference of potentials which exists
between the two insulated poles of a Daniell's cell.

We have imagined the rails to be connected by a wire
of very great resistance in comparison with that of the
rest of the circuit, and have supposed the length of this
wire to have remained constant. But from what we have
seen above, the effect of increasing the length of the wire,
the speed of the slider remaining the same, would be to
diminish the current in the ratio in which the resistance
is increased, and a correspondingly greater speed of the
slider would be necessary to maintain the current at the
same strength. We may therefore take the speed of the
slider as measuring the resistance of the wire. Now
suppose that when the slider 1 cm. long was moving at
the rate of 1 cm. per second, the current in the wire was
1 c.g.s. unit; the resistance of the wire was then 1 c.g.s.
unit of resistance. Unit resistance therefore corresponds
to a velocity of 1 cm. per second. This resistance, how-
ever, is too small to be practically useful, and a resistance
1,000,000,000 times as great, that is, the resistance of a
wire, to maintain 1 c.g.s. unit of current in which it would
be necessary that the slider should move with a velocity
of 1,000,000,000 cms. (approximately the length of a
quadrant of the earth from the equator to either pole) per
second, is taken as the practical unit of resistance, and
called one *ohm.*

In reducing the numerical expressions of physical quan-
tities from a system involving one set of fundamental units
to a system involving another set, as for instance from
the British foot-grain-second system, formerly in use for
the expression of magnetic quantities, to the c.g.s. system,

it is necessary to determine, according to the theory of "dimensions" first given by Fourier, and extended to electrical and magnetic quantities by Maxwell, for each a certain reducing factor, by substituting in the dimensional formula which states the relation of the fundamental units to one another in the expression of the quantity, the value of the units we are reducing from in terms of those we are reducing to. For example, in reducing a velocity say from miles per hour, to centimetres per second, we have to multiply the number expressing the velocity in the former units by the number of centimetres in a mile, and divide the product by the number of seconds in an hour ; that is, we have to multiply by the ratio of the number of centimetres in a mile to the number of seconds in an hour. The multiplier therefore, or *change-ratio* as it has been called by Professor James Thomson, is for velocity simply the number of the new units of velocity equivalent to one of the old units, and may be expressed by the formula $\frac{L}{T}$, where L is the number of new units of length contained in one of the old, and T is the corresponding number for the unit of time. In the same way the change-ratio for rate of change of velocity or acceleration is $\frac{L}{T^2}$; and the change-ratio of any other physical quantity may be found by determining from its definition the manner in which its unit involves the fundamental units of mass, length, and time. Now the theory of the dimensions of electrical and magnetic quantities, in the electromagnetic system of units, shows (ch. xii.) that the dimensional formula for resistance is the same as that for velocity ; that in fact a resistance in electro-magnetic measure is expressible as a velocity ; and hence we may

with propriety speak of a resistance of one ohm as a velocity of 10^9 centimetres per second.

The first experiments for the realisation of the ohm were made by the British Association Committee, and later determinations have been made with varying results by experimenters in different parts of the world. The method used by the committee was one suggested by Sir W. Thomson, in which the current induced in a coil of wire revolving round a vertical diameter in a magnetic field was measured by the deflection of a magnetic needle hung at the centre of the coil. The principle of the method is exactly the same as that of the ideal method, with slider and bars, sketched above. The committee found that according to their results the ohm is represented approximately by the resistance of a column of pure mercury 105 centimetres long, one square millimetre in section, at the temperature $0°$ C. Coils of an alloy of two parts of silver to one of platinum, which had a resistance of one ohm at a certain temperature, were issued by the committee as standards to experimenters.

Recent experiments by Lord Rayleigh and by Mr. Glasebrook at Cambridge have shown that the B.A. unit is probably about 1·3 per cent. too small. A still more recent and very careful determination by Lord Rayleigh and Mrs. H. Sidgwick gives ·98677 earth-quadrant per second as its value. A fresh determination is however being made by a committee of electricians appointed by the Paris Congress.

It is obvious from equation (14) that if V and R, each initially one unit, be increased in the same ratio, C will remain one unit of current; but that if V be, for example, 10^8 c.g.s. units of potential, or one volt, and R be a resistance of 10^9 cms. per second, or one ohm, C will be one-tenth of one c.g.s. unit of current. A current of

this strength—that is, the current flowing in a wire of resistance one ohm, between the two ends of which a difference of potentials of one volt is maintained,—has been adopted as the practical unit of current, and called one *ampere.* Hence it is to be remembered one ampere is one-tenth of one c.g.s. unit of current.

The amount of electricity conveyed in one second by a current of one ampere is called one *coulomb.* This unit, although not quite so frequently required as the others, is very useful, as, for instance, for expressing the quantities of electricity which a secondary cell is capable of yielding in various circumstances. For example, in comparing different cells with one another their capacities, or the total quantities of electricity they are capable of yielding when fully charged, are very conveniently reckoned in coulombs per square centimetre of the area across which the electrolytic action in each takes place.

The magneto-electric machine we have imagined gives us a very simple proof of the relation between the work done in maintaining a current, the strength of the current, and the electromotive force producing it. In dynamics *work* is said to be done *by* a force when its place of application has a component motion *in the direction in which the force acts,* and the work done by it is equal to the product of the force and the distance through which the place of application of the force has moved in that direction. The rate at which work is done by a force at any instant is therefore equal to the product of the force and the component of velocity in the direction of the force at that instant. The work done in overcoming a resistance through a certain distance is equal by this definition to the product of the resistance and the distance through which it is overcome. Among engineers in this country the unit of

work generally used, is *one foot-pound*, that is, the work
done in lifting a pound vertically against gravity through
a distance of one foot, and the unit rate of working is one
horse-power, that is 33,000 foot-pounds per minute. The
weight of a pound of matter being generally different at
different places, this unit of work is a variable one, and is
not used in theoretical dynamics. In the absolute c.g.s.
system of units, the unit of work is the work done in over-
coming a force of one dyne through a distance of one
centimetre, and is called one centimetre-dyne or one *erg*.
The single word *Activity* has been used by Sir Wm. Thom-
son as equivalent in meaning to "rate of doing work," or
the rate per unit of time at which energy is given out by
a working system; and to avoid circumlocutions in what
follows we shall frequently use the term in that sense.

We have seen above (p. 25) that every element of a
conductor carrying a current in a magnetic field is acted
on by a force tending to move it in a direction at right
angles to the plane through the element, and the direction
of the resultant magnetic force at the element, and have
derived from the expression for the magnitude of the
force a definition ((2) p. 25) of unit current in the electro-
magnetic system. From these considerations it follows
that a conductor in a uniform magnetic field, and carrying
a unit current which flows at right angles to the lines of
force, is acted on at every point by a force tending to move
it in a direction at right angles to its length, and the
magnitude of this force for unit length of conductor, and
unit field, is by the definition of unit current equal to unity.

Applying this to our slider in which we may suppose a
current of strength C to be kept flowing, say, from a
battery in the circuit, let L be the length of the slider,
v its velocity, and I the intensity of the field; we have

for the force on the moving conductor the value ILC.
Hence the rate at which work is done by the electro-
magnetic action between the current and the field is

$ILC\dfrac{dx}{dt}$ or $ILCv$, and this must be equal to the rate at

which work would be done in generating by motion of the
slider a current of strength C. But as we have seen above,
ILv is the electromotive force produced by the motion of
the slider. Calling this now E, the symbol usually em-
ployed to denote electromotive force, we have EC as the
electrical activity, that is, the total rate at which electrical
energy is given out in all forms in the circuit.

By Ohm's law this value for the electrical activity may

be put into either of the two other forms, namely, $\dfrac{E^2}{R}$, or

C^2R. In the latter of these forms the law was discovered
by Joule, who measured the amount of heat generated in
wires of different resistances by currents flowing through
them. This law holds for every electric circuit whether
of dynamo, battery, or thermoelectric arrangement.

We have, in what has gone before, supposed the slider
to have no resistance comparable with the whole resistance
in the circuit. If it has a resistance r, and R be the
remainder of the resistance in circuit, the actual difference
of potentials between its two ends will not be ILv or E,

but $E\dfrac{R}{R+r}$(p. 77). The rate per unit of time at which work

is given out in the circuit is however still EC, of which

the part $EC\dfrac{r}{R+r}$ is given out in the slider, and the

remainder, $EC\dfrac{R}{R+r}$, in the remainder of the circuit.

In short, if V be the actual difference of potentials, as

measured by an electrometer, between two points in a metallic wire connecting the terminals of a battery or dynamo, and C be the current flowing in the wire, the rate at which energy is given out is VC, or if R be the resistance of the wire between the two points, C^2R.

One of the great advantages of the system of units of which I have given this brief sketch, is that it states the value of the rate at which work is given out in the circuit, without its being necessary to introduce any coefficient such as would have been necessary if the units had been arbitrarily chosen. When the quantities are measured in c.g.s. units, the value of EC is given in centimetre-dynes, or in *ergs*, per second. Results thus expressed may be reduced to *horse-power* by dividing by the number $7{\cdot}46 \times 10^9$; or if E is measured in volts, and C in amperes, EC may be reduced to horse-power by dividing by 746. Thus, if 90 volts be maintained between the terminals of a pair of incandescent lamps joined in series, and a current of $1{\cdot}3$ ampere flows through these lamps, the rate at which energy is given out in the lamps is approximately $\cdot 157$ horse-power. If the rate at which work is done in maintaining a current of one ampere through a resistance of one ohm were taken as the practical unit of rate of working, or *activity*, and E reckoned in volts and C in amperes, the rate at which electrical energy is given out in the circuit would be measured simply by EC; and calculations of electrical work would be much simplified. This has been proposed by Sir William Siemens (Brit. Assoc. Address, 1882), who suggests that the name *watt* should be given to this unit rate of working. The rate at which energy is given out in the lamps of the above example is $90 \times 1{\cdot}3 = 117$ *watts*. A *watt* is therefore equivalent to 10^7 ergs per second.

Sir William Siemens has also proposed to call the work done in one second, when the rate of working is one watt, one *joule*.   Thus the work done in one second in maintaining a current of one ampere through one ohm, or the work obtained by letting down one coulomb of electricity through a difference of potentials of one volt, is one *joule*.   A *joule* is therefore equivalent to 10^7 ergs, and the work done in one second in the above example is 117 *joules*.

DESCRIPTION OF SIR WILLIAM THOMSON'S GRADED
GALVANOMETERS.

I. *The Potential Galvanometer.*

THE galvanometer used for measuring differences of
potential in electrical circuits is shown in Fig. 4 which
is engraved from a photograph of the actual instrument.
It consists of two essential parts, a coil and a magneto-
meter. The coil is generally made of silk covered
German silver wire of No. 32 B.W.G. When made
of German silver wire it contains about 2,200 yards of
wire wound in 7,000 turns, and has a resistance of over
6,000 ohms. It is made in the form of an anchor ring
having an outside diameter of fourteen centimetres and
an inside diameter of six centimetres. The diameter
of section is thus four centimetres. The coil is wound
within a mould of proper shape and dimensions, and
is then impregnated with melted paraffin under the
receiver of an air-pump. A solid compact ring is thus
obtained, which does not require a wooden case; and
which served round with a covering of silk ribbon looks well
and is not at all liable to get out of order. The coil thus
constructed is attached to one end of the horizontal
wooden platform P shown in the drawing, and kept firmly
in its place by a pair of wooden clamps fitted to the
lower half of the coil, and screwed firmly to the end of
the platform. When in position the plane of the coil is

vertical, and at right angles to a V groove that runs along the middle of the platform. The centre of the coil is
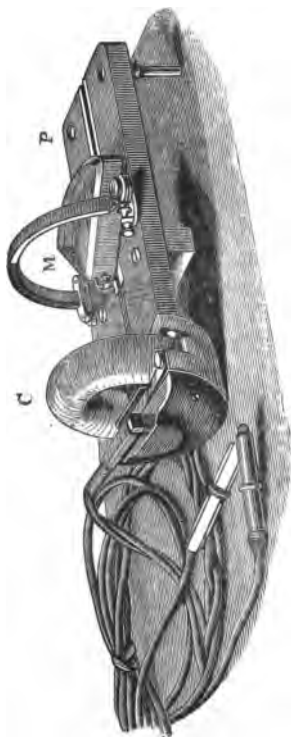
FIG. 4.

opposite to this groove and about one and a half centimetres above its bottom.

E 2

On the platform P rests the magnetometer M (shown in plan in Fig. 5), which consists essentially of a system of magnets properly supported so as to be free to turn round a vertical axis, and shielded from currents of air by being enclosed in a quadrantal shaped box having a closely fitting glass cover. Each magnet is fully one centimetre in length, and is made of glass-hard steel wire of No. 18 B.W.G. Four of these magnets mounted in a frame with their poles turned in similar directions form the "needle" of the instrument. The frame carrying the magnets is made of two thin bars of aluminium placed side by side with their planes vertical and about a centimetre apart, and connected by a bridge of sheet aluminium. The ends of the magnets are fixed in holes in the vertical sides of the aluminium frame so that the four steel needles form a set of four horizontal parallel edges of a rectangular prism. In the bridge connecting the two sides of the frame a sapphire cap is fixed, and this rests on an iridium-tipped point standing up from the bottom of the containing box. The sides of the frame are made long enough to form when brought together at one end an index, about nine centimetres long, of the shape shown in Fig. 2. The point of the index ranges round a scale of tangents placed round the curved edge of the bottom of the box. To prevent error from parallax the bottom of the box, with the exception of the narrow strip occupied by the scale, is covered with a mirror of silvered glass. The observer when taking a reading places his eye in such a position that the point of the index just covers its reflected image, and reads off the deflection indicated by the position of the point of the index on the scale of tangents. The scale is engraved on paper, and firmly fixed to the bottom of

the box by photographer's glue; and thus any change of length due to varying amount of moisture in the atmosphere is avoided.
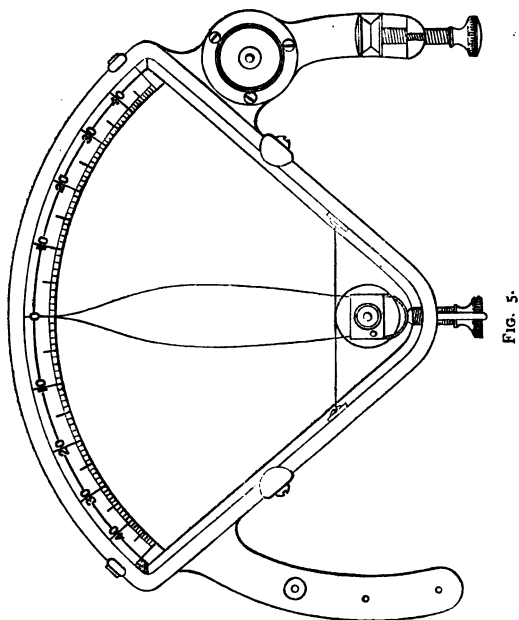
Fig. 5.

The magnetometer box rests on three feet and a flat spring. Two of these feet, which are in a plane perpendicular to the plane of the box and passing through the supporting point and the zero of the scale, slide in the v groove cut along the middle of the platform : the third foot rests on the plane surface on one side of this

groove, the spring on the other side. By this arrange-
ment the magnetometer is rendered perfectly steady and
can be moved with perfect freedom along, but only along
the platform. A small circular level carried by the box
shows when the plane of the magnetometer is horizontal.
This adjustment is made by means of the two screws which
support the platform at the end remote from the coil.

To lift the system of magnets and index off the bearing
cap when the instrument is not being used, or when it is
being carried from one place to another, a small collar-
piece free to move round the supporting point is raised
up by a horizontal screw turned by a head outside the
magnetometer box. When raised this collar-piece forms
a supporting platform for the needles and securely
prevents them from moving about and sustaining damage.

To increase the directive force on the needles when
required, the semicircular magnet shown in the drawing
is used with the instrument. This magnet is made of the
best steel, and is tempered glass-hard. It is magnetised by
sending a current through a semicircular coil containing it.
When in position on the instrument it is supported on
two flat pieces of brass projecting from the radial sides
of the magnetometer box. The magnet terminates at
one end in a cross-piece of brass having on its under
side at one end a small projecting brass knob. This
knob fits into a hollow in one of the projecting arms ·of
brass, while the other end of the cross-piece rests simply
on the plane surface of the arm. The other end of the
magnet is brought to a rounded point which rests
in a V notch cut round a cylindrical shoulder on a
screw spindle (seen on the right-hand side of Fig. 5),
which works through a nut fixed to the other projecting
arm of the magnetometer box. The magnet thus rests

with its magnetic axis as nearly as may be in the horizontal plane through the axis of the needle, and nearly at right angles to the line joining the centre of the needle's axis with the zero of the scale. Its axis may be placed accurately at right angles to this line by turning the screw until the needle points accurately to zero. The magnet thus mounted remains in the same position relatively to the magnetometer.

The coil is so adjusted that its centre is on a level with the magnetic axis of the needle when the magnetometer is in position. The centre of the magnetic axis, the zero of the scale, and the horizontal V groove in the platform are in the vertical plane through the centre of the coil. Hence if the magnetometer guided by its feet in the V groove be moved along the platform it will carry its magnet with it without disturbing the zero adjustment of the needle, and the magnets will in every position of the magnetometer be in the same field of force.

On the boxwood slip in which the V groove is cut is marked a series of positions of the front or circular edge of the magnetometer, for which the corresponding numbers of divisions of deflection for one volt difference of potentials, when the intensity of the magnetic field at the needle is one C.G.S. unit, are the terms of the geometric series . . . 8, 4, 2, 1, ½ . . . These numbers are stamped on the boxwood slip opposite the marks indicating the corresponding positions. The number of divisions of deflection for the nearest position of the magnetometer, that at which the centre of the magnetic axis of the needle is as nearly as may be at the centre of the coil, is not generally a term of this series, but it is determined in every case, and like the others is stamped on the platform.

The instrument is used for the measurement of high

potentials with the semicircular magnet in position ; but for low potentials the magnet is dispensed with, and the needle left under the earth's directive force alone. The field intensity given by the magnet of each instrument is determined before the instrument is sent out, and is painted on the magnet. The intensity of the field without the magnet, at the place at which the instrument is used, has, if necessary, to be determined. In practice it will generally be found convenient to use some position of the magnetometer which gives a convenient number of divisions of deflection per volt for the field employed. This position is determined by the user of the instrument, who marks it on the platform by drawing two vertical lines on the sides so as to prolong two white lines which are marked on the sides of the magnetometer.

The instrument as thus constructed admits of a very wide range of sensibility. By diminishing the distance of the magnetometer from the coil from the greatest to the least, the sensibility of the instrument can be increased fiftyfold : and by removing the field magnet from the instrument and leaving the needle under the influence of the earth's force alone, a sensibility fifty times still greater can be given to it. For the practical purposes for which these instruments are designed the suspension of the needle by cap and point is the most convenient ; but with this suspension there is always, with low directive forces, a slight error due to friction : and it is therefore not advisable to push the sensibility of the instrument further by diminishing the directive force of the earth's magnetism. An instrument of this kind, however, made for special purposes with a silk fibre suspension could be rendered more and more sensitive up to the limit of instability by so placing a magnet or magnets as, while

not interfering with the uniformity of the field at the needles, to diminish more and more the earth's directive force. This method of increasing the sensibility of a galvanometer although quite commonly used by scientific electricians is not, I have reason to believe, at all well-known generally, and recourse is had, altogether unnecessarily in many cases, to troublesome astatic combinations in order to obtain sensibility.

An important feature of this instrument in connection with its use for the measurement of high potentials is the arrangement of terminals which has been adopted. In certain circumstances when the ends of the coil of a potential galvanometer are attached to terminals fitted with binding screws, it is convenient to connect the instrument with the circuit by wires attached to these screws; but in the case of a dynamo circuit giving between the terminals of the coil a potential difference of eighty or a hundred volts and upwards, this plan of connections has been found highly dangerous. If the wires are twisted together and are ordinary gutta-percha-covered wires there is always a liability to accidents which may cause conduction from one wire to the other, and the destruction of the wires. Again, the ends of the wires are almost sure when removed from the instrument to be left dangling either in contact, or so as to be easily brought into contact inadvertently by a passer by, with the certain result, if the dynamo is running, of the immediate fusing of the wires. To prevent the possibility of such an accident Sir William Thomson has used as terminals for the coil two strong strips of copper about 1½ cm. broad which stand up vertically facing one another about a centimetre apart, within a vertical cavity in the wooden block behind the coil.

To prevent any current from flowing through the coil ex-
cept when a reading is being taken, the small spring contact
key, shown behind the coil in Fig. 4, is inserted between one
of these terminals and the coil.   The leads for connect-
ing the instrument with the circuit have their ends brought
together so as to terminate in two parallel strips of stout
copper kept apart by a piece of wood and held in posi-
tion by a good serving of strong waxed cord.   The two
copper strips with the piece of wood between them have
their ends turned down at right angles to their length,
and when connection is to be made are pushed down into
the cavity between the two bars to which the ends of
the coil are attached.   These bars are placed sufficiently
near together to be forced a little apart by the contact
piece, and thus give a secure spring contact. The leads are
made of thin stranded copper wire, well protected by a thick
woven covering of cotton, and are very flexible.   They ter-
minate in two spring clips (shown in Fig. 4) made each of a
strip of stout copper held firmly against the flat side of a
piece of wood, of semicircular section, by an india-rubber
band passed round a groove in a semicircular piece of
brass soldered to the copper strip, and round the back of
the piece of wood.   The copper strip lies in a groove
along part of the piece of wood, which prevents the strip
from turning round relatively to the wood, and thus a
good and safe contact is made between the copper and
anything on which it may be clipped.   These clips are
quite as efficient as binding screws and a great deal
more convenient.   They can in an instant be attached to
or removed from a wire or lead of any size and in any
position.   For convenience in the use of the instrument
the covering of one lead is coloured red, of the other
blue.

II. *The Current Galvanometer.*

This instrument is shown in Fig. 6. It differs from the galvanometer above described only in the coil and arrangement of terminals. The coil is made of a stout copper strip about 1·2 cm. broad and 1·5 mm. thick, wound in six turns, insulated by asbestos paper placed between. The outside diameter of the coil is about 10 cm., the inner diameter about 6 cm. It is covered like the other with silk ribbon, and attached in a similar manner to the platform P. A magnetometer exactly the same as that described above is used with the instrument, and all that has been said above with regard to the graduation of the potential galvanometer is applicable also in the present case, except that now amperes, not volts, are the subject of measurement.

This instrument is of course only suitable for the measurement of continuous currents, but owing to the small resistance of the coil, it can be left without risk of damage in a circuit with a current of upwards of 100 amperes flowing continually through it, while it is of sufficient sensibility to measure with accuracy, when the needle is acted on by the directive force of the earth alone, a current of from 1-10th to 1-100th of an ampere.

In special instruments for measuring very strong currents the coil is made of a single turn of massive copper strip, fitted with proper terminals to obviate undue heating at the contacts. With this mode of construction, an instrument can be made which shall measure with accuracy currents of from 1-10th of an ampere to 1000 amperes.

A pair of well-insulated leads several yards long, made of copper-wire cable containing 133 strands of wire of ·32 mm. diameter (No. 30 B.W.G.), and therefore very flexible and of inappreciable resistance, are sent out with

each instrument to be used with it. These are shown coiled on the table beside the instrument in Fig. 6.

The terminals of the instrument and the mode of including it, by means of its leads, in any circuit in which it is to be used, are worthy of a little attention. In order that the galvanometer may be used to measure the currents in different circuits, it must be introduced into or withdrawn from each circuit with as little disturbance as possible to the current in that circuit. To do this without the complications of switches or arrangements of binding screws, the very simple plan of terminals, shown in Fig. 3, has been adopted. The ends of the copper strip forming the coil are brought out horizontally behind the instrument, one above the other, with a thin piece of wood between them for insulator. On one end of the leads for attachment to these terminals is a spring clip, formed of two stout strips of copper, one attached to each lead, kept apart for a short distance along their length by a thickish piece of wood, and held in their places by a serving of waxed cord at that place. The ends of the copper strips project beyond the separating piece of wood about two or three inches, and are bent round into similar curves, with their convexities turned towards one another. They have sufficient spring to bring their convex portions into contact, but they are held together at that place by a stout india-rubber band passed round a groove in the edges of the two semicircular pieces soldered on the backs of the strips. The points, however, of the strips are a little distance apart. If, now, the clip just described be pushed over the terminals of the coil, the jaws of the clip will be separated, but before separation takes place each of them has come into contact with a terminal of the coil. Hence if the
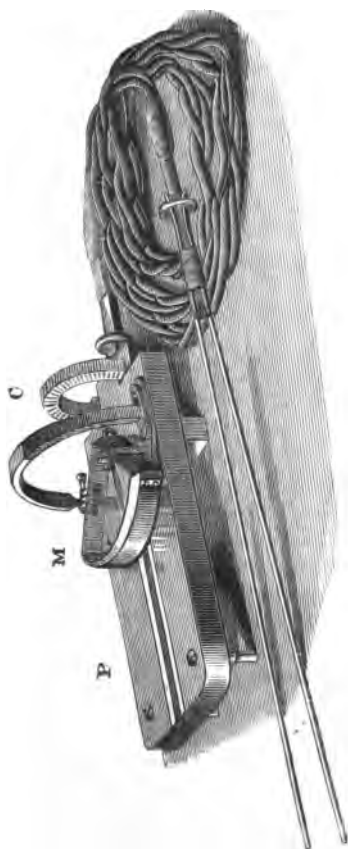
FIG. 6.

leads form part of a galvanic circuit, the current, before the galvanometer is attached, passes from one lead to the other across the jaws of the clips, and after these have been separated, through the galvanometer coil; and it is plain that no cessation of the current, and in practical cases only an infinitesimal disturbance can be caused by introducing the galvanometer. Sparks are thus altogether avoided, and the galvanometer is included in the circuit by a single simple and sure operation. When the leads are withdrawn from the coil-terminals the action is simply the reverse, the jaws of the clip have come together at their convexities before the terminals of the coil have lost contact with them.

In practice two stout wires which have one pair of ends attached to one of these spring clips are included in each circuit, the current in which is to be measured by the galvanometer. The instrument is placed with its leads attached to its terminals in a convenient position, so that the free end of the leads may reach easily the spring clips of all the circuits. The terminals at that end are similar to those of the galvanometer. They can therefore be pushed in between the jaws of each clip to allow the current to be read off and withdrawn without disturbing the current in the circuit. The leads are shown attached at one end by their spring clip to the galvanometer, and at the other end to a spring clip supposed included by means of the two straight pieces of wire in a galvanic circuit.

This arrangement is exceedingly useful for a great number of purposes, as for example for measuring the currents charging secondary cells, or flowing through the various parts of an electric lighting circuit or for measuring the whole current sent into the circuit by the dynamo or generator.

CHAPTER VII.

GRADUATION OF SIR WILLIAM THOMSON'S GRADED GALVANOMETERS.

WE shall now consider, very briefly, the graduation of instruments for measuring volts and amperes in practical work, and we shall take as our example Sir William Thomson's graded galvanometers. The graduation of these instruments is effected by a comparison of their indications with those of a standard galvanometer such as that described above. We shall consider first the graduation of a potential galvanometer, or galvanometer the resistance of which is so high, that the attachment of its terminals to two points in a conductor carrying a current does not perceptibly change the difference of potentials formerly existing between these points. Of course any galvanometer which measures currents also measures potentials, for, if its resistance is known, the difference of potentials between its terminals can be calculated from Ohm's law; but the convenience of a galvanometer specially made with a high resistance coil is that its terminals may be applied at any two points in a working circuit, and the difference of potentials, thus calculated as existing between these two points while the

terminals are in contact, may, in most cases, be taken as the actual difference of potentials which exists between the same points when nothing but the ordinary conductor connects them. For, let V be this actual difference of potentials in volts, let r ohms be the resistance of the conductor, and R ohms the resistance of the galvanometer. Then (p. 77) by the application of R, V is diminished in the ratio of R to $R + r$, and therefore the difference of potentials between the ends of the coil is now $V\dfrac{R}{R + r}$. Hence by Ohm's law we have for the current through the galvanometer the value $\dfrac{V}{R}\dfrac{R}{R + r}$, or $\dfrac{V}{R\left(1 + \dfrac{r}{R}\right)}$. If r be only a small fraction of R, $\dfrac{r}{R}$ is inappreciable, and the difference of potentials calculated from the equation $C = \dfrac{V}{R}$ will be nearly enough the true value.

The instrument to be graduated is first tested as to the adjustment of its coil, needle, &c. The standard galvanometer and it are then properly set up with their needles pointing to zero, in positions near which there is no iron, and at which the values of H have been determined. The high resistance coil of the standard galvanometer and the coil of the potential instrument are joined in series with a constant battery of as many Daniell's cells as gives a deflection of about 45° on the standard galvanometer, and the magnetometer is adjusted with its index at zero, in such a position on its platform that a deflection of its needle also of nearly 45° is produced. The current actually flowing in the circuit is calculated by equation (11) or (12) from the reading obtained on the

standard, and reduced to amperes by multiplying the result by 10. The difference of potentials between the two ends of the coil of the potential galvanometer is found in volts by multiplying the number of amperes thus found by the resistance of the coil in ohms. We can then obtain, by an obvious calculation, the number of divisions of deflection which corresponds to one volt between the two ends of the coil, and thence from the value of H the number of divisions which would corre-spond to one volt if the intensity of the field were one c.g.s. unit. This would be the number which would be marked at that position on the platform of the instru-ment; but, except for the position of the magnetometer nearest to the coil, positions the corresponding numbers of which are multiples and submultiples of 2 are alone marked. The numbers corresponding to the two of these positions adjacent on the two sides of the position of the magnetometer in our experiment, may be readily found by keeping the same difference of potential on the coil, and moving the magnetometer nearer to or further from the coil until the deflection is increased or diminished in the proper ratio. For example, let the deflection be 40 divisions for 20 volts, and let the value of H at the instru-ment which is being graduated be ·17. The number of divisions of deflection which would correspond to one volt for that position, if the field were of unit intensity, would be $\frac{40}{20} \times ·17 = ·34.$ Hence the marked positions nearest to this in the two sides of it, are to be those for which the corresponding numbers are $\frac{1}{2}$ and $\frac{1}{4}$. Therefore the magnetometer must be moved further from the coil for the latter until the deflection produced by 20 volts be-comes 29·4. This is the position at which the number $\frac{1}{4}$ is to be marked. To find the position at which $\frac{1}{2}$ should

be marked a smaller difference of potentials must be used, as the deflection with the same battery as before would be beyond the limits of the scale. Suppose that when its number of cells is diminished to one half; we get a deflection for our first position of 20. While the potential difference in the coil remains constant, the magnetometer is pushed in until the deflection again becomes 29·4. At this position the number ½ is to be marked. From this it is easy to see how the position for the number 1 can be found, and in the same way those for the other numbers of the series, 2, 4, &c. The number corresponding to the position of the magneto- meter nearest to the coil, although not one of the terms of this series, is determined in the same way, and marked at that position on the platform. This is the method adopted in practice in the graduation of these instru- ments.

Another method sometimes convenient is as follows. The standard instrument, a few good Daniell's cells, and a resistance which gives a deflection of about 45° on the standard are joined in series, and the galvanometer to be graduated is applied at two points in the circuit which include between them such a portion of the resistance as gives a deflection of about the same amount. Let R ohms be the portion of the resistance included between the terminals of the galvanometer, and let G ohms be the resistance of the galvanometer coil. Let the current calculated from the deflection on the standard be C amperes, then if V be the potential difference in volts between the terminals of the potential instrument, we have by Ohm's law—

$$C = \frac{V}{R},$$

where R' is the resistance equivalent to the divided circuit of R and G. But (p. 84) $R' = \dfrac{R\,G}{R + G}$, and therefore

$$C = V \frac{R + G}{R\,G}.$$

Hence,

$$V = C\,\frac{R\,G}{R + G}.$$

This last equation gives the number of volts indicated by the deflection on the potential instrument, for the position at which its magnetometer is placed; and from this in precisely the same manner as described above, the series of positions of the magnetometer on its platform are determined and numbered.

For verifying the accuracy of the graduation of the potential instruments when performed by either of these methods, a standard Daniell's cell of the form proposed by Sir William Thomson at the Southampton meeting of the British Association is used. It is represented in the annexed cut (Fig. 7). It consists of a zinc plate at the bottom of the vessel resting in a stratum of saturated zinc sulphate, on which has been poured, so gently as to give a clear surface of separation, a stratum of half saturated sulphate of copper solution, in which is immersed a horizontal plate of copper. The copper-sulphate solution is introduced by means of the glass tube shown in the diagram dipping down into the liquid, and terminating in a fine point, which is bent into a horizontal direction so as to deliver the liquid with as little disturbance as possible. This tube is connected by a piece of india-rubber tubing with a funnel, by the raising or lowering of which the sulphate of copper can be run into or run out of the cell. By this means the sulphate of copper is run in when the

cell is to be used, and at once removed when the cell is no longer wanted.

The electromotive force of this cell has been determined very carefully and found, according to Lord Rayleigh's latest determination of the ohm, to be 1·072 volt at ordinary temperatures. The direct application of this cell to the galvanometer gives at once a check on the graduation. As the resistance of the galvanometer is always over 6,000 ohms, there is practically no polarization.
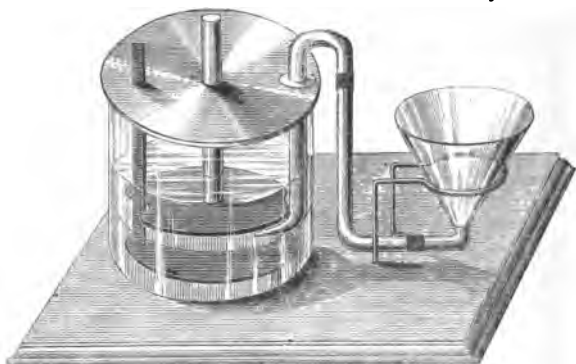
FIG. 7.

The standard cell is very conveniently used along with a Daniell's battery in the manner represented in the diagram, Fig. 8. *C* is the standard cell, and *B* a battery of from 30 to 40 small gravity Daniells.[1] A circuit is formed

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[1] These can be very easily made by using large preserve-pots as containing vessels, and placing at the bottom of each a copper disk of from three to three and a half inches in diameter, in a stratum of saturated copper sulphate solution, and a grating or plate of zinc a little below the mouth of the vessel immersed in a solution of zinc sulphate, of density 1·2. The copper sulphate may be kept saturated by crystals dropped into a glass tube passing down through a hole in the zinc plate to the copper. A copper wire well covered with gutta percha should be used as the electrode of the copper plate.

of a resistance box, the galvanometer G to be graduated, and the battery B joined in series with the standard cell C. A sensitive galvanometer D, which may be a reflecting galvanometer, or any very sensitive galvano-meter of low resistance, has one terminal attached at a point M between the battery and the standard cell, and the other terminal through the key K to an intermediate terminal L of the resistance box. The resistances in the box, on the two sides of L, are adjusted until no current flows through the galvanometer D, when the key K is depressed.
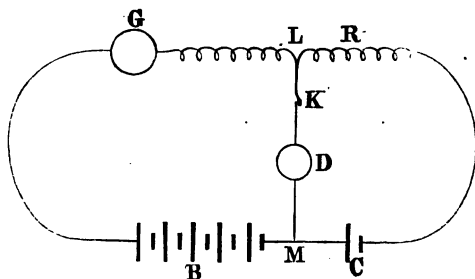
Fig. 8.

Let R be the resistance in the box to the right of L, r the resistance of the cell C, and G the resistance of the galvanometer. Then if V be the difference of potentials, in volts, between the terminals of the galvanometer,

$$V = 1 \cdot 072 \, \frac{G}{R + r}.$$

In practice a resistance of from 300 to 400 ohms is

generally required for R. The electromotive force of the
standard cell is taken as 1·072, as, notwithstanding the
large battery in the circuit, the total resistance is so.
great that there is very little polarization.

The magnetometer is placed so as to give a convenient
deflection with this known potential difference, and the
remainder of the graduation is proceeded with as
before.

Another method of using the standard cell for deter-
mining the difference of potentials given at the terminals
of the galvanometer G by the battery B is obtained by
placing R between L and M in Fig. 8, K and D in the
position occupied by R, and reversing the cell C. R is
adjusted until no current flows through C when the key
K is tapped down for an instant.   When this is the case
the electromotive force of C is balanced by the difference
of potentials at the two ends of R produced by B. Hence
the difference of potentials in volts then existing between
the terminals of G is given by the equation,

$$V = 1\cdot072 \, \frac{G}{R}.$$

By this method, which is an application of Poggendorff's
method of comparing the electromotive forces of batteries,
balance is obtained when no current is flowing through
the standard cell, and disturbance from polarization is
altogether avoided.   It has been found very easy and
convenient in practice.

The method adopted for the graduation of the current
galvanometer is precisely the same as that first described
for the potential instrument. The standard galvanometer,
of which in this case the low resistance coil is used, and
the current galvanometer to be graduated are joined in
series with a battery, which with some resistance in circuit

is sufficient to give a deflection in each of about 45°, when the magnetometer of the current instrument is at a convenient position on its platform. The current flowing in amperes is given by the standard, and this, of course, is the number of amperes which is indicated by the deflection of the current instrument. By an obvious calculation from the value of H at the current instrument, precisely similar to that above described, the number of divisions of deflection corresponding to a current of one ampere for a field of unit strength is found, and from this the series of positions on the platform and their numbers are found.

The value of the field intensity given by the magnet at the needles of the magnetometer when in position has generally been determined in the following manner. A battery of about 30 of Sir William Thomson's Tray Daniell's is joined in series with a resistance of about 7,000 ohms. The electrodes of a potential galvanometer, on which the magnetometer is placed without its magnet, are attached at two such points in this resistance that the deflection of the needle produced is from 30 to 40 division on the scale. The current through the galvanometer is now stopped and the magnet placed in position, and the index brought to zero by turning the magnet by means of its screw. The electrodes are now placed so as to include a resistance which makes the deflection nearly what it was in the former case. Let E be the electro-motive force of the battery, I the total horizontal intensity of the magnetic field at the needles when the magnet is in position ; R_1 R_2 the resistance included between the electrodes of the galvanometer in the first and second cases respectively ; V_1 and V_2 the potential difference in volts on the instrument in the same two cases ; D_1 and D_2

the corresponding deflections, and G the resistance of the galvanometer. We have by Ohm's law

$$V_1 = \frac{E R_1 G}{(B + R - R_1)(R_1 + G) + R_1 G} = m H D_1$$

and

$$V_2 = \frac{E R_2 G}{(B + R - R_2)(R_2 + G) + R_2 G} = m I D_2$$

where m is a constant. Therefore we have

$$I = H \frac{D_1 R_2 \left\{ (B + R - R_1)(R_1 + G) + R_1 G \right\}}{D_2 R_1 \left\{ (B + R - R_2)(R_2 + G) + R_2 G \right\}}$$

If the resistance B of the battery be small in comparison with G, or if the galvanometer is sensitive enough to allow $\frac{B}{G}$ to be made sufficiently small by resistance added to G, B may be neglected; and it is generally possible, by properly choosing R, R_1, R_2, to simplify very much this formula. The number I thus found, diminished by H, is the number of c.g.s. units which measures the horizontal intensity of the magnetic field produced at the needle by the action of the magnet alone. The value of $I - H$ is the number painted on the magnet, and is generally about 9 or 10.

The known potential obtained by comparing a Daniell's battery with the standard cell by the method described on pp. 69, 70, may be used to give a deflection with the magnet in position, and this deflection compared with that obtained at the same position of the magnetometer, with the earth's force alone, and the standard cell directly applied to the instrument. From this the ratio of I to H can obviously be at once obtained.

$I - H$ may be determined by means of a current galvanometer very easily by keeping a constant current flowing through the instrument, and using without the magnet

one of the less sensitive positions of the magnetometer, and with the magnet one of the more sensitive positions. If D_1 and D_2 be the deflections and $n_1\, n_2$ the number of divisions of deflections corresponding to one ampere for unit field at the two respective positions, the value of I is at once found from the obvious equation

$$\frac{D_1}{n_1 H} = \frac{D_2}{n_2 I}, \text{ or, } I = \frac{n_1 D_2}{n_2 D} H.$$

The value of I may be found at any place and for any magnet by means of the potential galvanometer, as follows. Place the magnetometer of the instrument at a convenient position, and apply a difference of potentials of V volts (measured by comparison with a standard cell as described in p. 70) to the terminals of the coil. Let the deflection produced be D scale-divisions, and let n be the platform number for the position of the magnetometer; then $I = \dfrac{n V}{D}$.

Denoting the electromotive force of the standard cell in volts by E, we have as in p. 70, $V = E \dfrac{G}{R}$, and therefore $I = \dfrac{n E G}{D R}$.

When either instrument has been graduated for a field of intensity equal to 1 c.g.s. unit, and the intensity of the field given by the magnet at the needles has been determined, the graduation of the instrument is complete. In the practical use of the instrument with the magnet in position, the number of volts, or the number of amperes (according as the instrument used is a potential or a current galvanometer) corresponding to a deflection of any number of divisions is found by the following rule:—

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1 This method is very convenient for testing the constancy of the field magnets from time to time. Such tests must of course be made with a reliable standard cell, and at a place remote from magnets and iron.

Multiply the number of divisions in the deflection by the number on the magnet increased by the horizontal intensity of the earth's field,[1] *and divide by the number at the division on the platform scale exactly under the front of the magnetometer.*

When the magnet is not used the rule is the same as the above, except that the divisor to be used is the value of H for the place of the galvanometer.

For convenience in the ordinary use of either instrument a position of the magnetometer on its platform, which may not be one of the series described above, is determined at which the deflection with the magnet in position, for one volt or one ampere, is one or some other convenient number of divisions. For this position lines are drawn on the sides of the platform so as to prolong the white lines on the sides of the magnetometer. By this means currents in amperes or potentials in volts can be at once read off without any calculation.

The graduation of the current galvanometer may also be performed by means of electrolysis. The electro-chemical equivalents of a large number of the metals have been determined, and it is only necessary therefore, in order to graduate the instrument, to join it in series with a proper electrolytic cell and a constant battery, and to compare the amount of metal deposited on the negative plate of the cell with the total quantity of electricity which flows through the circuit in a certain time. A convenient cell may be made with two plates of copper, each about 50 square cms. in area, held parallel to one another about 2 cms. apart, and immersed in a nearly neutral solution of copper-sulphate in a glass beaker. A

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[1] The mean value of H for Great Britain may, *when the magnet is used*, be taken with sufficient accuracy as ·17 c.g.s.

current of one ampere through such a cell gives a very good
result. This current will deposit about 1·2 grammes of
copper per hour, which, when light plates are used, can
be very accurately weighed. The plates should be care-
fully cleaned, dried, and weighed before being immersed
in the liquid, and care must be taken not to send the
current through the cell at all until it is made to flow
continuously for the experiment. The instant the current
through the cell is completed should be noted, and
readings of the current galvanometer taken at equal
short intervals during the time allowed for the experi-
ment. The average of these readings is to be taken as
the average reading of the galvanometer. When the
experiment has been completed the plates are to be
taken out and very carefully washed in clean water and
dried before being weighed. It is generally better to
calculate the quantity of electricity from the gain of the
negative plate than from the loss of the positive. The
electro-chemical equivalent of copper has been recently
determined afresh with great care in the Physical
Laboratory of the University of Glasgow by Mr. Thomas
Gray, and as the result of his experiments ·000331 of a
gramme of copper is deposited by the passage of one
coulomb of electricity. Dividing the gain of weight of
the negative plate by this number, we obtain the total
number of coulombs which have flowed through the
galvanometer during the experiment. This divided by
the time in seconds gives the average current in amperes ;
from which the number corresponding to the position of
the magnetometer on the platform can be determined,
and the graduation completed in the manner described
above.

CHAPTER VIII.

POTENTIALS AND CURRENTS IN DERIVED CIRCUITS.

IN Chapter V. above we have given a short explanation of
Ohm's law; in the present chapter we shall consider
this law a little more at length, and show how to derive
from it rules for the calculation of the equivalent resist-
ance of any arrangement of derived circuits, and the
strength of the current in any part of that arrangement.
Suppose that we have an electric generator E, the two
terminals of which are joined by a single copper wire.
Then by the statement of Ohm's law above, if R be the
resistance of the wire between any two points in itself,
and V the difference of potentials maintained between
those two points, the strength C of the current flowing in
the wire is given by the equation $C = \dfrac{V}{R}$. Now it is a
fact proved by experiment that, if a current of constant
strength C be kept flowing across any cross-section of the
circuit of E, the value of C is the same throughout the
whole circuit. Hence if we take the points A and B
nearer to the terminals of the generator, and thus
increase the resistance R of the copper wire intercepted
between them, V must increase in the same ratio to give
the same value of C as before. Thus the difference of

potentials between any two equipotential surfaces in a homogeneous conductor anywhere in the circuit of a generator giving a constant current, may be proved to be proportional to the resistance of the conductor between these two surfaces. Now let us apply this conclusion to the case of the circuit of a voltaic cell composed of two dissimilar plates of metal immersed in a liquid, for example copper and zinc immersed in dilute sulphuric acid, and connected externally by a wire W.

Let C and Z denote the copper and zinc plates, L the liquid between them. There will, by the theory of the voltaic cell now generally adopted, be certain finite differences of potential on the two sides of the junction of the dissimilar metals, and on the two sides of the junction of each metal with the liquid. We may suppose the plates to be such that they add no sensible resistance to the circuit. Let V_a denote the potential of the copper plate; V_b the potential of the copper wire close to its junction with the zinc plate; V_{lz} the potential of the stratum of the liquid close to the zinc plate; and V_{lc} the potential of the stratum of the liquid close to the copper plate. The difference of potentials between two points in the copper conductor near its ends is therefore $V_a - V_b$, and that between the two sides of the liquid $V_{lz} - V_{lc}$. Both of these differences must be taken as positive, since the current flows from the copper plate to the zinc in the external part of the circuit, and from the zinc plate to the copper within the liquid. Calling R the resistance of the conductor joining the plates, and r the resistance of the liquid of the cell, we have, since the current has the same strength in every part of the circuit,

$$C = \frac{V_a - V_b}{R} = \frac{V_{lz} - V_{lc}}{r} \quad . \quad . \quad . \quad (1)$$

and therefore also,

$$C = \frac{V_a - V_{lc} + V_{lz} - V_b}{R + r} \quad . \quad . \quad . \quad (2)$$

But $V_a - V_{lc}$ is the finite difference between the potential of the copper plate and that of the liquid in contact with it, and $V_{lz} - V_b$ is similarly the difference between the potential of the liquid in contact with the zinc plate and that of the zinc plate itself, and the sum of these two differences constitutes what is called the electromotive force of the cell.   Calling this E we have

$$C = \frac{E}{R + r}. \quad . \quad . \quad . \quad . \quad (3)$$

If V be the difference of potentials between any two points in the copper wire, R the resistance of the wire between these two points, and r the remainder of the resistance in the circuit, we have from the equations,

$$C = \frac{V}{R} = \frac{E}{R + r}$$

the result,

$$V = E \frac{R}{R + r} \quad . \quad . \quad . \quad . \quad (4)$$

the value used in p. 64.

  The value of E in this equation varies to some extent with the strength of the current in all batteries, being generally greater for weak than for strong currents, and a maximum when the cell is on open circuit.   The variation is due in most cases to the deposition of a film of gas on one or both of the plates which produces an electromotive force opposed to that of the cell, and generally also offers resistance to the current.   The diminution of the value of E thus produced is what is properly called *polarization*.   In all one-fluid batteries the effect of polarization is very marked, but in two-fluid batteries such as those of Daniell, Grove, and Bunsen,

in which the evolution of hydrogen gas at the negative electrode is in great measure prevented, its amount, if the current be not too great, is very small. In the large surface tray Daniells of Sir William Thomson the value of E, even when a strong current is flowing through the battery, is sensibly the same as when the battery is on open circuit; and with but little attention from the operator this battery will, if the solution of copper sulphate is kept saturated, yield a strong and constant current for days together.

We have considered our electrical generator as one cell of a voltaic battery, but, if it consist of more cells than one, its electromotive force is found in exactly the same manner, by summing all the discontinuities or sudden differences of potential in the circuit. Hence if there be n cells in the battery, joined in series, that is to say the zinc plate of the first cell joined to the copper plate of the second cell, the zinc of the second to the copper plate of the third, and so on to the last cell, the total electromotive force of the arrangement, if the cells have all the same electromotive force, and E be that of the cell, will be $n E$. If the copper plate of the first cell and the zinc plate of the last be joined by a wire, and R denote the total external resistance in circuit, r as before the internal resistance of each cell, a current of strength C given by the equation,

$$C = \frac{n E}{R + n r}$$

will flow in the wire. This equation may be written,

$$C = \frac{E}{r + \dfrac{R}{n}} \quad \cdot \quad \cdot \quad \quad \cdot \quad (5)$$

which shows that when n is so great that $\dfrac{R}{n}$ is small in

comparison with r, little is added to the value of C by further increasing the number of cells in the battery.

The method of joining single cells in series is advantageous when R is large, but when R is comparatively small it fails as shown above, and it is necessary then to diminish r as much as possible. The value of r is, for cells in which, as is generally the case, each plate nearly covers the cross-section of the liquid, nearly in the inverse ratio of the area of the plates, and directly as the distance between them. Hence by increasing the area of the plates and placing them as close together as possible, the resistance may be diminished. One large cell of small resistance may be formed of several small cells by putting all the copper plates into metallic connection with one another, and similarly all the zinc plates. Several compound cells of large surface thus made may be joined in series. The electromotive force of each compound cell will be E as in a simple cell, but if m cells be joined so as to form one compound cell its resistance will be $\frac{r}{m}$. If n of these compound cells be joined in series, we have, calling the total external resistance R,

$$C = \frac{n E}{R + n \dfrac{r}{m}} = \frac{m n E}{m R + n r} \quad \cdot \quad \cdot \quad \cdot \quad (6)$$

If R be not too great, and we have a proper number of cells, it is possible to arrange the battery so that C may have a maximum value. There being $m n$ cells in the battery the numerator of the above value of C does not change when the arrangement of cells is varied, and therefore in order that C may have its greatest possible value $m R + n r$ must be made as small as possible. But identically,

$$m R + n r = (\sqrt{m R} - \sqrt{n r})^2 + 2 \sqrt{m n R r}$$

As the last term on the right-hand side does not vary with the arrangement of the battery, it is plain that $m R + n r$ will have its smallest value when $\sqrt{m R} - \sqrt{n r}$ vanishes, that is when $m R = n r$ or $R = n \dfrac{r}{m}$, or, in words, when the total external resistance of the circuit is equal to the internal resistance of the battery. It may not be possible in practice so to join a given battery as to fulfil this condition, but if the strongest possible current is required it should be fulfilled as nearly as possible. This method of arranging the battery is called joining it in *multiple arc*.

It is to be carefully observed that this theorem applies only to the case in which we have a given battery and have to arrange it so as to produce the *greatest current* through a given external resistance R; and the fallacy is to be avoided of supposing that of two batteries of equal electromotive force, but one having a high, the other a low, resistance, the former is better adapted for working through a high external resistance. Nor is this method of using the battery to be confounded with the most *economical* method. By this arrangement the energy of the battery is most rapidly expended, as much being given out in the battery itself as in the external resistance, and it is plain that for economy as little as possible of the energy of the battery must be spent in the battery itself, and as much as possible in the working part of the circuit. Hence for economical working the internal resistance of the battery and the resistance of the wires connecting the battery with the working part of the circuit must be made as small as possible. We shall return to this question in Chapter X.

The electromotive force of a magneto- or dynamo-electric

machine, that is, a machine in which an electromotive force is produced by the motion of wires across the lines of force of a magnetic field, depends on the length of wire so moving in the field, the intensity of the field, and the speed at which the armature is caused to revolve. Such a machine producing a continuous current differs from a voltaic cell in the fact that there is a gradual rise of potentials from the negative terminal towards the positive along each portion of wire moving across the lines of force of the field (supposed uniform), and a gradual fall along every other portion of the armature, the sum of which constitutes the difference of potentials between the terminals; while in the cell the electromotive force consists in sudden changes of potential at the surfaces of the heterogeneous surfaces in contact. When the magnetic field of the machine is kept unaltered, the available difference of potentials at the two ends of any conducting wire, joining the terminals of such a machine, is to be found by Ohm's law in precisely the same way as for a voltaic battery, and all that has been said above as to the arranging of cells in series or in multiple arc, is applicable also in this case.

Although Ohm's law holds for all cases of conduction when everything affecting electromotive force or resistance is taken into account, the formulas given above are not applicable to self-exciting machines, or dynamos as they are called, in which the whole current or a portion of the current generated by the machine itself is employed to magnetize the field magnets; for here the magnetization of the magnets, and consequently the total electromotive force of the machine at a given speed, is dependent on the resistances in circuit.

An exposition of the theory of such machines is beyond

the scope of the present work, and is the less necessary as by means of regulators, or by the process of double winding the dynamo so as to combine the principle of the type of machine in which the electromagnet coils are in a derived circuit, with that of the type in which they are placed directly· in the main circuit, the difference of potentials between the terminals can be kept nearly constant over a considerable range of resistances in the working circuit.   We shall therefore give here only a few useful rules for the calculation of currents and resistances in derived circuits connected with a machine between the terminals of which a constant difference of potentials is maintained.
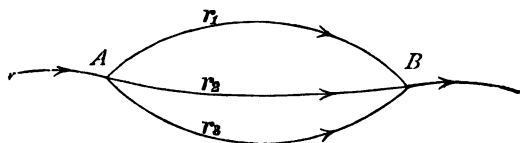
Fig. 9.

It has been stated above that when a constant current is kept flowing across any cross-section of a circuit, composed of an electrical generator with its poles joined by a single conductor, the current strength is the same across every other section of the circuit ; or, in other words, that the flow of electricity at any instant into any portion of the conductor is equal to the flow out of the same portion. This is what is called the *principle of continuity* as applied to the case of a *steady* flow of electricity.   By the same principle we have in the case in which steady currents are maintained in the various parts of a network of conductors the theorem, that the total flow of electricity towards the point at which several wires meet is

equal to the total flow from that point. Thus the current arriving at A (Fig.9) by the main conductor is equal to the sum of the currents which flow from A by the arcs r_1, r_2, &c., which connect it with B. This theorem is generally given as *Kirchhoff's first law*.

By Ohm's law (p. 40) if two points A and B between which a difference of potentials V is maintained be connected by two wires of resistances r_1 and r_2 the current in that of resistance r_1 will be $\dfrac{V}{r_1}$ and in the other $\dfrac{V}{r_2}$. But if C be the whole current flowing in the circuit we have by Kirchhoff's law,

$$C = \frac{V}{r_1} + \frac{V}{r_2} = \frac{V}{R} \quad . \quad . \quad . \quad . \quad (7)$$

where R is the resistance of a wire which might be substituted for the double arc between A and B without altering the current in the circuit. Hence,

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{R} \quad . \quad . \quad . \quad . \quad (8)$$

and

$$R = \frac{r_1 r_2}{r_1 + r_2} \quad . \quad . \quad . \quad . \quad (9)$$

The reciprocal of the resistance R of a wire, that is, $\dfrac{1}{R}$, is called its *conductivity*. Equation (8) therefore affirms that the conductivity of a wire, the substitution of which for r_1 and r_2 between A and B, would not affect the current in the circuit, is equal to the sum of the conductivities of the wires r_1 and r_2. From equation (9) we see that the resistance R of this equivalent wire is equal to the product of the resistances of the two wires divided by their sum.

If for r_2 we were to substitute two wires having an equivalent resistance, so that A and B should be connected,

as in Fig. 9, by three separate wires of resistances r_1, r_2, r_3, we should have in the same manner for the current in

$r_1, \dfrac{V}{r_1}$; in $r_2, \dfrac{V}{r_2}$; in $r_3, \dfrac{V}{r_3}$, and

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \quad . \quad . \quad . \quad . \quad (10)$$

and therefore

$$R = \frac{r_1 \, r_2 \, r_3}{r_1 \, r_2 + r_2 \, r_3 + r_3 \, r_4} \quad . \quad . \quad (11)$$

Generally, if two points A and B are connected by a multiple arc consisting of n separate wires, the conductivity of the wire equivalent to the multiple arc connection is equal to the sum of the conductivities of the n connecting wires ; and its resistance is equal to the product of the n resistances divided by the sum of all the *different* products which can be formed from the n resistances by taking them $n - 1$ at a time.

As a simple example we may take the case of a number n of incandescence lamps joined in multiple arc. If the resistance of each lamp and its connections be r, the equivalent resistance between the main conductors, the resistance due to the latter being neglected, is by (11) $\dfrac{r^n}{n \, r^{n-1}} = \dfrac{r}{n}$. Thus if r be 60 ohms when the lamp is incandescent, and there be twenty lamps, their resistance to the current will be 3 ohms.

By the considerations stated above, we at once deduce from Ohm's law a second theorem generally given as *Kirchhoff's second law*. In any closed circuit of conductors forming part of any linear system, the sum of the products obtained by multiplying the current in each part taken in order round the circuit by its resistance is equal to the sum of the electromotive forces in the circuit.

This follows at once by an application of Ohm's law to each part of the circuit, exactly as in the investigation in p. 27 above of the electromotive force of the circuit composed of a cell and a single conductor. As an example of a circuit containing no electromotive forces, consider the circuit formed by the two
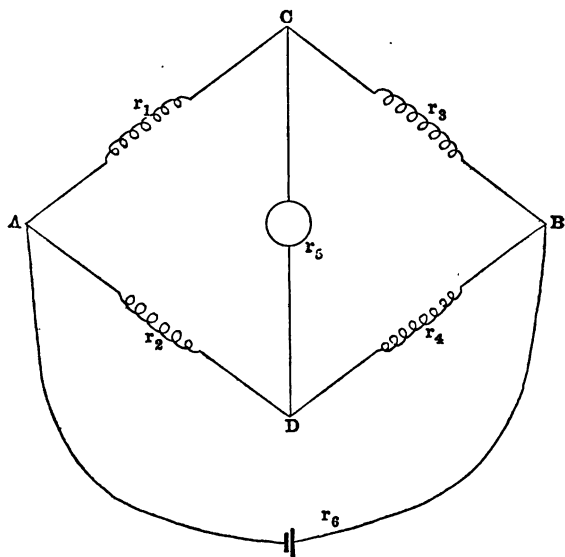
Fig. 10.

wires of resistance r_1, r_2 joining $A\,B$. We have for the current flowing from A to B through r, the value $\dfrac{V}{r_1}$; the product of this by r, is V; for the current flowing

from B to A through r_2 we have $-\dfrac{V}{r_2}$, and the product of this by r_2 is $-V$; the sum is $V - V$ or zero. As another example consider the diagram Fig. 10 of resistances r_1, r_2, r_3, r_4, r_5, between the two points A and B. By what we have seen if V_a, V_b, V_c, V_d, be the potentials at A, B, C, D respectively, the current from A to C is $\dfrac{V_a - V_c}{r_1}$, from C to D $\dfrac{V_c - V_d}{r_5}$, and from D to A $\dfrac{V_d - V_a}{r_2}$. Hence taking the sum of each of these current strengths by the corresponding resistances for the circuit $A\,C\,D\,A$ we get

$$V_a - V_c + V_c - V_d + V_d - V_a = o.$$

The following reciprocal relation between any two conductors A_{lm}, A_{pq} of a linear system is sometimes useful. An electromotive force which placed in A_{lm} causes a current to flow in A_{pq}, would if placed in A_{pq} cause an equal current to flow in A_{lm}. For a proof of this proposition the student may refer to Maxwell's *Electricity and Magnetism*, vol. I, p. 335 (first edition).[1]

As an example of the use of Kirchhoff's laws we may apply them to find the current strength in r_5 when the terminals of a battery of electromotive force E are applied at A and B. Let r_6 be the resistance of the battery and the wires connecting it with A and B, and let γ_1, γ_2, &c., be the strengths of the currents flowing in the resistances r_1, r_2, &c. respectively, in the directions indicated by the arrows. By Kirchhoff's first law we get,

$$\left. \begin{array}{l} \gamma_3 = \gamma_1 - \gamma_5 \\ \gamma_4 = \gamma_2 + \gamma_5 \\ \gamma_5 = \gamma_1 + \gamma_2 \end{array} \right\} \quad \cdot \ \cdot \ \cdot \ \cdot \ \cdot \quad (12)$$

Applying the second law to the circuits $BACB$, $ACDA$,

[1] The list of *Errata* to the first edition contains an important correction in the statement of this theorem.

$CBDC$, and using equations (12) we obtain the three equations,

$$\left.\begin{array}{l} \gamma_1 (r_1 + r_3 + r_6) + \gamma_2 r_6 - \gamma_5 r_3 \qquad\qquad = E \\ \gamma_1 r_1 \qquad\quad - \gamma_2 r_2 + \gamma_5 r_5 \qquad\qquad = 0 \\ \gamma_1 r_3 \qquad\quad - \gamma_2 r_4 - \gamma_5 (r_3 + r_4 + r_6) = 0 \end{array}\right\} (13)$$

Eliminating γ_1 and γ_2, we find,

$$\gamma_5 = \frac{E (r_2 r_3 - r_1 r_4)}{D} \quad \ldots \quad (14)$$

where

$$D = r_5 r_6 (r_1 + r_2 + r_3 + r_4) + r_5 (r_1 + r_3) (r_2 + r_4)$$
$$+ r_6 (r_1 + r_2) (r_3 + r_4) + r_1 r_3 (r_2 + r_4) + r_2 r_4 (r_1 + r_3)$$

In order that γ_5 may be equal to zero, we have the condition,

$$r_2 r_3 - r_1 r_4 = 0 \quad \ldots \quad (15)$$

By substituting for γ_2 in the second and third of equations (13) its value $\gamma_6 - \gamma_1$, we get,

$$\left.\begin{array}{l} \gamma_1 (r_1 + r_2) + \gamma_5 r_5 \qquad\qquad - \gamma_6 r_2 = 0 \\ \gamma_1 (r_3 + r_4) - \gamma_5 (r_3 + r_4 + r_5) - \gamma_6 r_4 = 0 \end{array}\right\} (16)$$

From these we obtain by eliminating γ_1,

$$\gamma_5 = \frac{\gamma_6 (r_2 r_3 - r_1 r_4)}{r_5 (r_1 + r_2 + r_3 + r_4) + (r_1 + r_2)(r_3 + r_4)} \cdot (17)$$

By means of equations (14) and (17) we can very easily solve the problem of finding the equivalent resistance of the system of five resistances r_1, r_2, &c., between A and B. For let R be this equivalent resistance, since γ_6 is the current flowing through the battery, we have $\gamma_6 = \dfrac{E}{r_6 + R}$. Substituting this value of γ_6 in (17), equating the values of γ_5 given by (14) and (17), and solving for R, we get,

$$R = \frac{r_5 (r_1 + r_3)(r_2 + r_4) + r_1 r_3 (r_2 + r_4) + r_2 r_4 (r_1 + r_3)}{r_5 (r_1 + r_2 + r_3 + r_4) + (r_1 + r_2)(r_3 + r_4)} (18)$$

CHAPTER IX.

THE COMPARISON OF RESISTANCES.

THE arrangement of resistances shown in Fig. 10 is
that used in the method of comparing the resist-
ances of conductors, known as Wheatstone's Bridge. A
battery, generally a single Daniell's or Menotti's cell is
sufficient, is placed in r_6 as indicated in the figure ; and
a sensitive galvanometer (preferably a dead-beat mirror
galvanometer of one or two ohms resistance) is included
in r_5. The galvanometer is, of course, set up so that the
needle hangs, when the current is zero, parallel to the plane
of the coil, and the spot of light is therefore near the middle
of the scale. Generally r_1, r_2, r_3 are coils of a resistance
box provided with terminals so arranged that connections
can be made at the proper places to form the bridge. If
the connecting wires of the bridge pass near the galvano-
meter, they are twisted together so that currents in them
can produce no direct effect on the needle. The resistance
to be measured is placed in the position r_4, and convenient
values of r_1 and r_2 are chosen, while r_3 is varied until no
current flows through the galvanometer. The value of r_4
is then found by equation (15), which may be written,

$$r_4 = \frac{r_2}{r_1} r_3 . \quad . \quad . \quad . \quad . \quad (1)$$

If r_1 and r_2 are equal, r_4 is equal to r_3, and is read off at
once from the resistance box.

In the practical use of Wheatstone's Bridge, we have generally to employ a certain battery, and a certain galvanometer for the measurement of a wide range of resistances; and it is possible if great accuracy is required so to choose the resistances of the bridge as to make the arrangement have maximum sensibility. An approximate determination is first made of the resistance to be measured. Call this r_4. It has been shown by Mr. Oliver Heaviside (*Phil. Mag.*, vol. 45, 1873), and by Mr. Thomas Gray (*Phil. Mag.*, vol 12, 1881), that we should make,

$$r_1 = \sqrt{r_5 r_6}, \; r_3 = \sqrt{r_4 r_6 \frac{r_4 + r_5}{r_4 + r_6}}, \; r_2 = \sqrt{r_4 r_5 \frac{r_4 + r_6}{r_4 + r_5}}$$

In general, however, with resistance boxes as usually constructed, it is not possible to make this adjustment except very roughly. This is of the less importance as throughout a wide range of values of r_4, any convenient values of r_1, r_2 will give results sufficiently accurate for all practical purposes; but in arranging the bridge with these the following rule should be observed: of the resistances r_5, r_6 of the galvanometer and battery respectively, connect the greater so as to join the junction of the two greatest of the four other resistances to the junction of the two least.

This rule follows easily from equation (14) of Chap. VII. above. For, interchanging r_5 and r_6, we alter only the value of D, and calling the new value D' we get,

$$D' - D = (r_5 - r_6)(r_1 - r_4)(r_3 - r_2).$$

This expression on the right will be negative if $r_6 > r_5$ and r_1, r_3 be the two greatest or the two least of the other resistances. Hence on this supposition the value of D has been diminished, and therefore the current through the galvanometer for any small value of $r_2 \, r_3 - r_6 \, r_4$ diminished by making r_6 join the junction of r_1, r_3 to that of r_2, r_4.

The galvanometer should also be made as sensitive as possible by diminishing the directive force on the needle as far as is practicable without rendering the needle unstable. This is easily done by placing magnets near the coil so that the needle hangs, when the current in the coil is zero, in a very weak magnetic field. That the field has been weakened, by any change in the disposition of the magnets made in the course of the adjustment, will be shown by a lengthening of the period of free vibration of the needle when deflected for an instant by a magnet and allowed to return to zero. The limit of instability has been reached when the position of the spot of light for zero current changes from place to place on the scale, and the intensity of the field must then be slightly raised to make the zero position of the needle one of stable equilibrium.

Although not absolutely essential, except when accurate readings of deflections are required, it is always well, when the field is produced by magnets, to arrange them so that the field at the needle is nearly uniform. It may therefore be produced by two or more long magnets placed parallel to one another at a little distance apart symmetrically with respect to the centre of the needle above or below it, and with their like poles turned in the same directions, or a long magnet placed horizontally with its centre over the needle, and mounted on a vertical rod so that it can be slided up or down to give the required sensibility may be used.

In the practical use of this method, a spring contact key, which makes contact only when depressed, should be placed in r_5, and another in r_6. These keys are conveniently arranged side by side. The key in r_6 which completes the battery circuit, is first depressed by the operator, and immediately after the key in r_5 is also

depressed. After the circuits have been completed just long enough to enable the operator to see whether there is any deflection of the needle, the keys are released so as to break the contacts in the reverse order to that in which they were made. The operator may easily imagine and construct a form of contact-making key, which being depressed a certain distance completes the battery circuit, and on being depressed a little further completes the galvanometer circuit, and therefore on being released interrupts these circuits in the reverse order.

The object of thus completing and interrupting the battery circuit first is, partly, to avoid error from the effects of *self-induction*. When a current in a conducting wire is being increased or diminished, an electromotive force, the amount of which depends on the arrangement of the conductor, is called into play so as to oppose the increase or diminution of the current. The effect of this electromotive force is to produce therefore a weakening of the electromotive force of a battery when the circuit is completed, and a strengthening at the instant the circuit is interrupted. Its value is zero when the wire is doubled on itself, so that the two parts lie along side by side, the current flowing out in one and back in the other ; but is very considerable if the wire is wound in a helix, and is still greater if the helix contains an iron core. The electromotive force of self-induction is directly proportional to the rate of variation of the current in the circuit, and thus is explained the bright spark seen when the circuit of a powerful electromagnet is *broken*.

If then one or more of the coils of a Wheatstone Bridge arrangement are wound so as to have self-induction, the electromotive force thus called into play would, if the galvanometer circuit were completed before that of the

battery, produce a sudden deflection of the galvanometer needle when the battery circuit is closed. All properly constructed resistance coils are made of wires which have been first doubled on themselves and then wound double on their bobbins, and have therefore no self-induction. The wire tested, however, and the connections of the bridge have generally more or less self-induction, the effect of which, unless the contacts were made as described above, might be mistaken by the operator for those of an unbalanced resistance.

By winding the coils in this manner, also, any direct electromagnetic effect on the needles due to the current in the coils is rendered impossible, and the galvanometer may, without risk of disturbance, be placed comparatively near the resistance box.

When comparing a resistance, the operator first observes the direction in which the mirror or needle is deflected when a value of r_3 obviously too great is used, and again when a much smaller value of r_3 is used. If the deflections are in opposite directions, the value of r_3, which would produce no deflection of the needle, lies between these two values, and the operator simply narrows the limits of r_3, until on depressing the galvanometer key, no motion, or only a very small motion, of the needle is produced. It may happen, however, that the value of the resistance which is being compared may lie between two resistances which have the smallest difference which the box allows. Thus with a resistance box, by which with equal values of r_1 and r_2 he cannot measure to less than $\frac{1}{10}$ of an ohm, he may either by making the ratio of r_1 to r_2, 10 to 1, or 100 to 1, obtain the values of r_4 to one or to two more places of decimals. Or, whatever be the ratio of r_1 to r_4, if he can read the deflections when

first one and then the other value of r_3, between which r_4 lies, and which differ by only $\frac{1}{10}$ of an ohm, is used, he can find r_4 to a further place of decimals by interpolation. For example, let the value 120·6 of r_3 produce a deflection of the spot of light of 6 divisions to the left, and 120·5 a deflection of 14 divisions to the right, the value of r_3 which would produce balance is equal to

$$120\cdot5 + \frac{\cdot1}{14+6} \times 14 = 120\cdot57.$$

Resistances may however be compared with greater
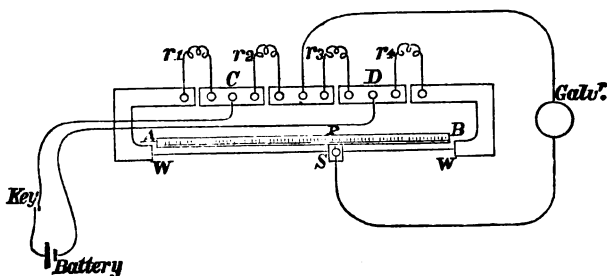
accuracy by means of Kirchhoff's modification of Wheatstone's Bridge, in which an exact balance is obtained by moving a sliding contact piece along a graduated wire which joins the two resistances r_1, r_4 of Fig. 11. This method was used by Messrs. Matthiessen and Hockin in the very careful comparisons of resistances made by them in their work, as members of the British Association Committee ; and it was found by these experimenters that an alloy of 85 parts of platinum with 15 parts of iridium formed an excellent material for the graduated wire. This alloy they found did not readily become oxidized,

did not change much in conducting power with alteration of temperature, and did not alloy with mercury. Fig. 11. shows the arrangements of a bridge on this plan, $W W$ is the platinum-iridium wire soldered at its two ends to thick copper plates which connect it in series with the four resistances r_1, r_2, r_3, r_4, of which we may take r_4 as the resistance to be compared. One terminal of the battery is connected through a key with a binding screw between r_2 and r_3, the other to a spring contact attached to the sliding piece S, which, as it is moved along the wire, carries an index along the graduated scale $A B$. The method of testing by this instrument is precisely the same as by the ordinary Wheatstone Bridge, except that when balance has been nearly obtained in the usual way, by varying the relation of the resistances r_1, r_2, r_3, for a particular position of the sliding piece, an exact balance is obtained by shifting the sliding piece in the proper direction along the wire. Supposing that the resistance of the wire per unit of length has been determined for different parts of the wire, and that the resistances of contacts have been determined (p. 104 below) and allowed for, the value of r_4 is at once found by taking into account the resistances of the segments of the wire $W W$ on the two sides of the point contact at which gives zero deflection.

The wire $W W$ may be "calibrated" by the following method, which was that employed by Matthiessen and Hockin.[1] Let r_1 and r_4 be coils of such resistances, that balance is obtained at some point P in $W W$, and let r_2, r_3 be two coils, differing in resistance by say $\frac{1}{10}$ per cent. Let $r_1 + a$ be the total resistance, including contacts between C and P, and $r_4 + \beta$ that between D and P. Now alter r_1 by inserting a short piece of wire. This

[1] Reports on Electrical Standards, p. 119.

will shift the zero point along the wire through a certain distance to the right. Balance so as to find this point, which call P_1; then interchange r_2 and r_3, and balance again, and call the second point thus found P_2. Let z denote the resistance between P and P_1, z' the resistance between P and P_2, x the resistance of the short piece of wire added to r_2, and l the length of wire between P_1 and P_2. We have plainly the two equations,

$$\left. \begin{array}{c} \dfrac{r_1 + a + x + z}{r_2} = \dfrac{r_4 + \beta - z}{r_3} \\[2mm] \dfrac{r_1 + a + x + z'}{r_3} = \dfrac{r_4 + \beta + z'}{r_2} \end{array} \right\} \quad \cdot \quad (2)$$

From which we obtain for the resistance per unit of length between P_1 and P_2,

$$\frac{z - z'}{l} = \frac{r_2 - r_3}{l\,(r_2 + r_3)}\,(r_1 + r_4 + a + \beta + x) \cdot \quad (3)$$

The value of x is easily obtained with sufficient accuracy from either of equations (2), as z is approximately known from the known resistance of the whole wire. In this way the resistance per unit of length at different parts of the wire can be easily found, and, if necessary, a table of corrections formed for the different divisions of the scale.

Sir William Thomson has shown how to measure the resistance of a galvanometer by means of Wheatstone's Bridge, without the use of a second instrument. The galvanometer is placed in the position of r_4 and a spring contact key in r_5. If when the key is depressed no current flow in r_5, the current through the galvanometer will be unaltered; r_1, r_2, r_3, are therefore adjusted until the current through the galvanometer caused by completing the battery circuit is not altered by depressing the key in r_5. The resistance r_4 of the galvanometer and its connections is then given by equation (1) above.

Since the needle of the galvanometer is in the most sensitive position when at zero, that is parallel to the plane of the coil, a deflection opposite and nearly equal to that produced by depressing the battery key should be given to it by a controlling magnet placed at right angles to the plane of the coil at the proper distance. When the key is depressed, the needle will come nearly to the zero position, and will there be most sensitive to any change of the current produced by depressing the key in r_5.

Neither of the arrangements of Wheatstone's Bridge described above is at all suitable for the comparison of the resistances of short pieces of thick wire or rod, for example, specimens of the main conductors of a low resistance electric light installation, the resistances of which are so small as to be comparable with, if not less than, the resistances of the contacts of the different wires by which they are joined for measurement. To obtain an accurate result in such a case as this, we must compare the difference of potentials between two cross-sections in the rod which is being tested, with the difference of potentials between two cross-sections in a standard rod, while the same current flows in both rods, at and everywhere between each pair of cross-sections, in a direction parallel to the axis. Sir William Thomson has so modified Wheatstone's Bridge, by adding to it what he has called *secondary conductors*, as to enable it to be used, with all the convenience of the ordinary arrangement, for the accurate comparison of the resistance of a foot or two of thick copper conductor with that between two cross-sections in a standard rod. The arrangement is shown in Fig 13. $C D$ are two cross-sections, at a little distance from the ends of the conductor to be tested, and $A B$ are two similar cross-sections of the standard conductor.

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These rods are connected by a thick piece of metal, so that the resistance between B and C is very small, and the terminals of a battery of low resistance are applied at the other extremities of the rods as shown. The sections $B C$ are connected also by a wire $B L C$, and the sections $A D$ by a wire $A M D$, in each case by as good metallic contacts as possible. $B L C$ and $A M D$ may very conveniently be wires, along which sliding contact-pieces L and M can be moved, with resistances R, R, R, R of half an ohm or an ohm each, inserted as shown in the figure. The
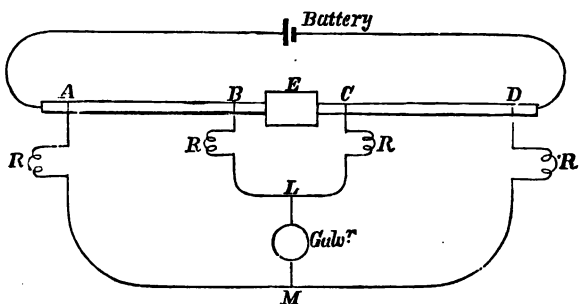
FIG. 12.

sections A, D are so far from the ends of the rods, and the wires $A M D$, $B L C$ are made of so great resistance (one or two ohms is enough in most cases), that the current throughout the portions of the conductors compared is parallel to the axis, and the effect of any small resistance of contact there may be at A, B, C, D is simply to increase the effective resistance of B C and $A M D$ by a small fraction of the actual resistance of the wire in each case. The terminals of the galvanometer G are

applied at L and M, and the circuits of the galvanometer and battery are completed through a double key as in the ordinary bridge. Let the resistances AM, DM be denoted by r_1, r_3; BL, CL by a, b, AB, CD by r_2. r_4 and BC by s. Suppose r_1 and r_3 to be varied by moving the sliding piece at M till no current flows through the galvanometer. To find the relation which must hold among the resistances when this is the case, we may suppose the point L connected by a bar of zero resistance, with the cross-section of E, which is at the same potential as L. Call this cross-section K. The resistance of the portion of BC to the left of K is, by equation (1), $\dfrac{a\,s}{a+b}$ and of the portion to the right $\dfrac{b\,s}{a+b}$. The resistance between B and KL is by equation (9, p. 84) above $\dfrac{a\,\dfrac{a\,s}{a+b}}{a+\dfrac{a\,s}{a+b}}$ or

$\dfrac{a\,s}{a+b+s}$, and similarly that between C and KL is $\dfrac{b\,s}{a+b+s}$. Hence again by equation (1) we have,

$$r_3\left(r_2+\frac{a\,s}{a+b+s}\right)=r_1\left(r_4+\frac{b\,s}{a+b+s}\right)$$

or,

$$r_1\,r_4-r_3\,r_4=\frac{s}{a+b+s}\,(ar_3-br_1)\quad.\quad(4)$$

Now s has been supposed very small in comparison with $a+b$, and a and b can be easily chosen so as to make ar_3-br_1 approximately equal to zero. Hence equation (4) reduces to,

$$r_4=\frac{r_3}{r_4}r_2\quad.\quad.\quad.\quad.\quad.\quad.\quad(5)$$

the formula found above for the ordinary Wheatstone
Bridge.

The following method was used for the same purpose by
Messrs. Matthiessen and Hockin in their researches on
alloys. $A B, C D$, Fig. 12, are the two rods to be compared.
They are connected in circuit with two resistance coils
$R R'$, which have between them a graduated wire $W W'$,
as in Kirchhoff's bridge. $S S'$ are two sharp knife-edges,
the distance of which apart can be accurately measured,
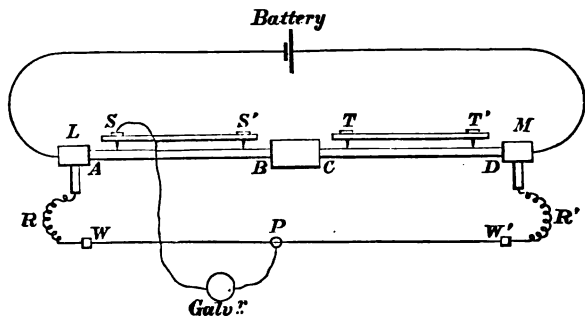
FIG. 13.

fixed in a piece of dry hard wood or vulcanite, and con-
nected with mercury cups on its upper side. This
arrangement is placed on the conductor $A B$, so that the
knife-edges making contact include between them a length
$S S'$ of the rod. $T T'$ is a precisely similar arrangement
placed on $C D$. One terminal of the galvanometer is
applied at S, and the resistances $R R'$ adjusted so that a
point P_1 on the wire which gives balance is found for the
other terminal. The terminal of the galvanometer is

shifted to S', and a second point P_2 found by varying R, R' in such a manner as to keep $R + R'$ constant. Similarly balance is found for $T\,T'$.   Let $L\,S, L\,S'$ denote the resistances between L and S, L and S', &c., respectively ; $W\,P_1$, $W'\,P_1$ the resistances between W and P, and W' and P in the first case, and so on.   We have,

$$\frac{S\,L}{S\,M} = \frac{R_1 + W\,P_1}{R'_1 + W'\,P_1} \; ;$$

and therefore,

$$\frac{L\,S}{L\,M} = \frac{R_1 + W\,P_1}{R},$$

where

$$R = R_1 + R'_1 + W\,W'.$$

Similarly,

$$\frac{L\,S'}{L\,M} = \frac{R_2 + W\,P_2}{R},$$

and

$$\frac{S\,S'}{L\,M} = \frac{R_2 - R_1 + P_1\,P_2}{R}.$$

In the same way we get,

$$\frac{T\,T'}{L\,M} = \frac{R_4 - R_3 + P_3\,P_4}{R}$$

and combining the last two equations we get for the ratic of the resistances of the conductors between the pairs of knife-edges—

$$\frac{S\,S'}{T\,T'} = \frac{R_2 - R_1 + P_1\,P_2}{R_4 - R_3 + P_3\,P_4}. \quad . \quad . \quad (6)$$

The method of comparing resistances mentioned in p. 3, is in principle the same as Thomson's Bridge with secondary conductors, and Matthiessen and Hockin's method described above, as, like them, it consists in comparing the difference of potentials between two cross-sections near the ends of the conductor to be

tested with the difference of potentials between two cross-sections in a standard conductor, when the same uniform current is flowing in both. It is however more readily applicable in practice, and is very useful for a great many purposes, as for example in the testing of the armatures or magnet coils of machines, in the estimation of the resistances of contacts, and in the determination of the specific conductivities of thick copper wires or rods. All that is required is a small battery, a high resistance galvanometer of sufficient sensibility, and two or three resistance coils of from $\frac{1}{8}$ ohm to 1 ohm. These coils may very conveniently for many purposes be made of galvanised or tinned iron wire of No. 14 or 16 B.W.G., wound round a piece of wood $\frac{1}{2}$ inch thick, from 8 to 10 inches broad, and from 12 to 18 inches long, with notches cut in its sides, at intervals of a quarter of an inch, to keep the wire in position. To avoid any electromagnetic effect which may be produced by the coils if they happen, when carrying currents, to be placed near the galvanometer, the wire should be doubled on itself at its middle point, the bight put round a pin fixed near one end of the board, and the wire then wound double on the board, the two parts being kept far enough apart to insure insulation. Resistance coils made in this way are exceedingly useful for electric-lighting experiments, as the thickness of the wire and its exposure everywhere to the air prevent undue heating by strong currents, or, if there is much heating, obviate the risk of damage. For the battery a single cell, as for example the standard cell described above, or if the battery is to be carried from place to place, two hermetically sealed chloride of silver cells, which may be joined in series or in multiple arc as required, may very conveniently be used. Sir William Thomson's graded

potential galvanometer is the most convenient instrument for most practical purposes, but when very great accuracy is aimed at, as when the method is used for the measurement of the specific conductivity of short lengths of thick metallic wires by comparison with a standard, a high resistance reflecting galvanometer should be employed, and in these circumstances the battery should be of as low internal resistance as possible.

The galvanometer is first set up and made of the requisite sensibility either by adjusting, as described in p. 91 above, the intensity of the field in which it is placed, or, if it is a graded galvanometer, by placing the magnetometer at the position nearest to the coil, · and dispensing with the field magnet.

The conductor whose resistance is to be compared, and one of the coils whose resistance is known, are joined in series with the battery.   It is advisable to have this circuit at a distance of a few yards from the galvanometer, so that accidental motions of the wires carrying the current may not have any sensible effect on the needle.   One operator then holds the electrodes of the galvanometer so as to include between them, say, first the wire which is being tested, then the known resistance, then once more the wire being tested, in every case taking care not to include any binding screw connection, or other contact of the conductors.

Let the mean of the readings for the first and third operations be V scale divisions, for the second V'; let r denote the known resistance, and x the resistance to be found.

Since by Ohm's law the difference of potentials between any two points in a homogeneous wire, forming part of a circuit in which a uniform current is flowing, is proportional

to the resistance between those two points, we have,

$$x = \frac{V'}{V} r. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The resistance of a contact of two wires whether or not of the same metal may be found in the same manner, by placing the galvanometer electrodes so as to include the contact between them, and comparing the difference of potential on its two sides with that between the two ends of a known resistance in the same circuit. Care must however be taken in all experiments made by this method, especially when the galvanometer circuit includes conductors of different metals, to make sure that no error is caused by thermal electromotive forces. To eliminate such errors the observations should be made with the current flowing first in one direction, and then in the other in the battery circuit.

The following results of some measurements of the resistance of a Siemens $S\,D_2$ dynamo machine, made on May 4, 1883, in the Physical Laboratory of the University of Glasgow, may serve to illustrate this method. An iron wire coil, of half an ohm resistance, was joined to one of the terminals of a standard Daniell, and short wires attached to the other terminal of the cell and the free end of the coil were made to complete the circuit through the armature, by being pressed on two diametrically opposite commutator bars, from which the brushes and the magnet connections had been removed. The electrodes of the galvanometer, which was one of Sir William Thomson's dead-beat reflecting galvanometers of high resistance, were applied alternately to the same commutator bars and to the ends of the half ohm, and the readings recorded. The following are the results, extracted from the Laboratory Records, of three consecutive experiments.

EXPERIMENT I.

Operation.	Reading on Scale.	Deflection of Spot of Light.
Galv. zero read	214	
Electrodes on ½ ohm	857	633
,, ,, armature	597	383

EXPERIMENT II.

	Reading on Scale.	Deflection of Spot of Light.
Galv. zero read	214	
Electrodes on armature	607	393
,, ,, ¼ ohm	874	660
,, ,, armature	607	393

EXPERIMENT III.

	Reading on Scale.	Deflection of Spot of Light.
Galv. zero read	214	
Electrodes on ½ ohm	874	660
,, ,, armature	607	393
,, ,, ¼ ohm	872	658

The first experiment gives for x the value, $\dfrac{383}{633} \times \cdot 5$, or

·298 ohm. The other two experiments, although their numbers are different, give exactly the same result, which agrees closely with a measurement made about eight months before, by the same method, with a graded potential galvanometer.

In the ordinary testing of the armatures of machines by this method, the circuit of the battery may be completed through the brushes; but if the machine has been wound on the shunt system, care must be taken to previously disconnect the magnet coils; and also in performing the experiments to place the galvanometer electrodes on the commutator bars directly.

In order that the conducting powers of different sub-stances may be compared with one another, it is necessary to determine their *specific resistances*, that is, the resist-

ance in each case, which a wire of a certain length and cross-sectional area, has at a certain temperature. We shall here define the specific resistance of any substance as the resistance between two opposite faces of a centimetre cube of the material at the temperature o° C. This resistance has been very carefully determined for several different substances by various experimenters, and a table of results is given below (Table V.).

To measure the specific resistance of a piece of thin wire, we have simply to determine the resistance of a sufficiently long piece of the wire by the ordinary Wheatstone Bridge method described above, and from the result to calculate the specific resistance. Let the length of the wire be l cms., its cross-section s square cms., and the resistance at o° C. calculated from the observed resistance (see Table III.) R ohms. Then the specific resistance of the material would be $\dfrac{Rs}{l}$ ohms. The length l is to be carefully determined by an accurately graduated measuring rod; and the area s may be found with sufficient accuracy in most cases by direct measurement, by means of a decimal wire gauge, measuring to a hundredth of a millimetre. If, however, the wire be very thin, the cross-section may, if the density is known, be accurately obtained in square cms. by finding the weight in grammes of a sufficiently long piece of the wire (from which the insulating covering, if any, has been carefully removed), and dividing the weight by the product of the length and the density. If the density is not known, it may be found by weighing the wire in air and in water by the methods described in books on hydrostatics.[1] Very thin wires are generally covered

1 For a clear statement of the method and of precautions to be observed, the reader may refer to Bottomley's *Hydrostatics* (Glasgow : Collins, Sons, d Co., 1882).

with silk or cotton, which may very easily be removed, without injury to the wire, by making the wire into a coil, and gently heating it in a dilute solution of caustic soda or potash. The coating must not in any case be removed by scraping.

If the wire be thick, and a sufficient length of it to render possible an accurate measurement of its resistance by the ordinary bridge method is not conveniently available, one of the methods of comparing small resistances described above (pp. 97–106) is to be used. The most convenient in many cases is that just described (pp. 101–106) in which the resistance between two cross-sections of the bar to be tested is compared with that between two cross-sections of a standard rod of pure copper. If we put V for the number of divisions of deflection on the scale of the potential galvanometer, when the electrodes of the galvanometer are applied to the standard rod, at cross-sections l cms apart ; V' that when they are applied to the rod being tested, at cross-sections l' cms. apart, then we have for the ratio of the resistance of unit length of the standard to the resistance of unit length of the wire tested at the temperature at which the comparison is made, the value $\dfrac{V\,l'}{V'\,l}$. If s and s' be the respective cross-sectional areas, which in this case are easily determinable by measurement with a screw gauge, and we assume that the temperature at which the measurements of resistance are made is $0°$ C., we get for the ratio of the specific resistances the value $\dfrac{V\,l'\,s}{V'\,l\,s'}$, and therefore also for the ratio of their specific conductivities $\dfrac{V'\,l\,s'}{V\,l'\,s}$. This last ratio multiplied by 100 gives the percentage conductivity of the substance

as compared with that of pure copper. If, as will generally be the case, the temperature at which the experiments are made be above the freezing point, the value of $100 \dfrac{V'\,l\,s}{V\,l'\,s'}$ may be taken as the percentage of the specific conductivity of pure copper at the observed temperature possessed by the substance, and this, if the wire tested is a specimen of nearly pure copper will be nearly enough the same at all ordinary temperatures.

If in experiments by this method the electrodes are applied by hand to the conductors, the operator should, especially if the electrodes and the conductors tested are of different materials, be careful not to handle the wires, but should hold them by two pieces of wood in strips of paper passed several times round the wires, or by some other substance which conducts heat badly, so that no thermal electromotive force may be introduced into the circuit of the galvanometer (see above, p. 104). He may conveniently make the galvanometer contacts by means of two knife edges fixed in a piece of wood which can be lifted from one conductor to the other without its being necessary to handle the galvanometer wires in any way. This will besides render any measurement of the length of the conductor intercepted between the galvanometer electrodes unnecessary, as l is equal to l'. We have then for the percentage specific conductivity of the substance the value $100 \dfrac{V'\,s'}{V\,s}$.

As an example of this method we may take the following results of a measurement (made in the Physical Laboratory of the University of Glasgow) of the specific conductivity of a short piece of thick copper strip. The specimen was joined in series with a piece of copper wire of No. o B.W.G., of very high conductivity, in the circuit

of a Daniell's cell of low resistance.   The electrodes of a
high resistance reflecting galvanometer applied at two
points 700 cms. apart in the copper wire gave a deflection
of 153·5 divisions, when applied at two points 500 cms.
apart in the strip 270 divisions.   The weight of the wire
per metre was 443 grammes, of the strip per metre
186·3 grammes.   Hence the specific conductivity of the
copper strip was 96·6 per cent of that of the wire against
which it was tested.

The measurement of a very high resistance such as that
of a piece of insulating material cannot be effected by
means of Wheatstone's Bridge, and recourse must be had
in most cases to electrostatic methods in which the required
resistance is deduced from the rate of loss of charge of a
condenser, the plates of which are connected by the
substance in question.   If, however, the resistance of the
material be not too great, and a large well insulated
battery of from 100 to 200 cells, and a very sensitive
high resistance galvanometer are available, the following
method is the most convenient.   First join the galvano-
meter, also well insulated, and the resistance to be
measured (prepared as described below, p. 113, to
prevent leakage) in series with as many cells as gives
a readable deflection, which call D.   Now join the
battery in series with the galvanometer alone, and
reduce the sensibility of the instrument to a suitable
degree by joining its terminals by a wire of known
resistance, and, to keep the total resistance in circuit
great in comparison with the resistance of the battery,
insert resistance in the circuit.   Let E and B denote
respectively the electromotive force and resistance of the
whole battery, G the resistance of the galvanometer, S
the resistance joining its terminals in the second case,

R the resistance introduced into the circuit of the galvano-meter in that case, and X the resistance to be found; we have for the difference of potentials between the terminals of the galvanometer in the first case the value,

$$\frac{E\ G}{G + B + X} = m\ D$$

where m is the factor by which the indications of the galvanometer must be multiplied to reduce them to volts. In the second case the resistance between the galvano-meter terminals is $\frac{S\ G}{S + G}$, and therefore the difference of potentials between them is,

$$\frac{E\ \dfrac{S\ G}{S + G}}{B + R + \dfrac{S\ G}{S + G}} = \frac{E\ S\ G}{(B + R)\,(S + G) + S\ G} = m\ D$$

Hence combining these equations so as to eliminate E and m, and solving for X, we get,

$$X = \frac{D_1}{D}\left(B + R + G + \frac{(B + R)\ G}{S}\right) - (B + G)\quad (8)$$

If X be great in comparison with the remainder of the resistance in circuit the term $(B + G)$ may be neglected.

A modification of this method for which Sir W. Thomson's Graded Potential Galvanometer is very suitable, may be used for the determination of the insulation resistance of the conductors in an electric light installation.

The conductors are disconnected from the generator and both ends from one another. They are then joined at one end by the potential galvanometer in series with a battery of as many cells as gives a readable deflection with the magnetometer in the position of the greatest sensibility. The number of volts corresponding to this deflection is read off, and the number of volts which the

battery gives when applied to the galvanometer alone is then observed. Call the latter number V and the former V'; and let E volts be the total electromotive force of the battery. Let the resistance of the battery which may be determined by the method described below (p. 121) be B ohms, the resistance of the galvanometer G ohms, and the insulation resistance to be found R ohms; we have plainly,

$$V = \frac{E\,G}{B + G}, \quad V' = \frac{E\,G}{B + G + R}.$$

Therefore,

$$R = (B + G)\left(\frac{V}{V'} - 1\right) \quad . \quad . \quad (9)$$

If B be small in comparison with G we may put

$$R = G\,\frac{V - V'}{V'}. \quad . \quad . \quad . \quad (10)$$

A shunt-wound generating machine giving sufficient electromotive force may be used instead of the battery, and in this case R is found by equation (10).

The insulation resistance per unit of length is found from this result by *multiplying* by the length of either of the conductors.

This method is applicable to the measurement of the insulation resistance of cables or telegraph lines, but for details the reader is referred to the manuals of testing in connection with these special subjects.

In the case of insulating substances the method just described requires the use of so powerful a battery that it is quite inapplicable except when the specimen, the resistance of which is to be measured, can be made to have a large surface perpendicular to the direction of the current through it, and of very small dimensions in that direction. Such a case is that of the insulating covering of a submarine cable in which the current by which

the insulation-resistance is measured flows across the covering between the copper conductor and the salt water in which the cable is immersed.

In general, therefore, in the determination of the insulating qualities of substances which are given in comparatively small specimens it is necessary to have recourse to the electrometer method mentioned above (p. 109), of which we shall give here a short account.

The most convenient instrument for this purpose is Sir William Thomson's Quadrant Electrometer. For a full description of this instrument, and a detailed account of the mode of using it, the reader is referred to *Reports on Electrical Standards*, or to *Reprint of Papers on Electrostatics and Magnetism*, by Sir W. Thomson (pp. 262-281). The electrometer, having been carefully set up according to the most sensitive arrangement, and found to be otherwise in good working order, is tested for insulation. One pair of quadrants is connected to the case according to the instructions for the use of the instrument, and a charge producing a potential difference exceeding the greatest to be used in the experiments is given to the insulated pair by means of a battery, one electrode of which is connected for an instant to the electrometer-case, the other at the same time to the electrode of the insulated quadrants, and the percentage fall of potentials produced in thirty minutes or an hour by leakage in the instrument is observed. If this is inappreciable, the instrument is in perfect order.

An air condenser, well insulated by glass stems varnished and kept dry by pumice moistened with strong sulphuric acid, is adjusted to have a considerable capacity, and its insulated plate is connected to the insulated quadrants of the electrometer and the other to the electrometer-case to

which the other pair of quadrants is also connected. A charge producing as great a potential as in the former case is given to the condenser and electrometer thus arranged, and the fall of potentials observed by means of the electrometer. If the loss in a considerable time be also inappreciable, the condenser insulates properly, and its resistance may be taken as infinite.[1]

The specimen of material to be tested is now placed so as to connect the plates of the condenser. The manner in which this is to be done of course depends on the form of the specimen. If it is a flat sheet, it may be covered on each side, with the exception of a wide margin all round, with tinfoil, and thus made to form itself a small condenser which is to be joined by thin wires in multiple arc with the large condenser. The edges and margins of the sides of the specimen should be carefully cleaned and dried, and covered with a thin coating of paraffin to prevent conduction along the surface between the two tinfoil coatings, when the condenser is charged. It is advisable, when possible, to coat the whole surface including the tinfoil with paraffin, and to make the contacts with the tinfoil plates by means of thin wires also coated with paraffin for some distance along their length from the tinfoil.

If the specimen be cup-shaped, as, for example, if it be in the usual form of an insulator for telegraph or other wires, the hollow may be partially filled with mercury, and the cup immersed in an outer vessel containing mercury, so that the mercury stands at nearly the same

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[1] A condenser of any other kind, such as those used in cable testing, the insulating material between the plates of which is generally paper soaked in paraffin, may be used instead of an air condenser, but as the resistance of the latter may, if the glass stems be well varnished and kept dry, be taken as infinite, and there is besides no disturbance from the phenomenon called *eectrification*, it is preferable to use an air condenser if possible.

level outside and inside. The lip of the cup down to the mercury on both sides is to be cleaned and coated with paraffin, as before, to prevent leakage across the surface. A thin wire connected with the insulated plate of the condenser is made to dip into the mercury in the cup, and a similar wire connected with the other plate of the condenser dips into the mercury in the outer vessel. Strong sulphuric acid may, on account of its drying properties, be used with advantage instead of mercury as here described, when the substance is not porous and is not attacked by the acid.

In every case in which, as in these, the insulating substance and the conductors making contact with it form a condenser of unknown capacity, the condenser used in the experiment must be arranged to have a capacity so great that the unknown capacity thus added to it, together with the capacity of the electrometer, may be neglected in the calculations.

The condenser, if it has been disconnected, is again connected as before to the electrometer. One electrode of a battery of from six to ten small Daniell's cells in good order, is also connected with the electrometer case, and the other electrode is brought for a short time, thirty seconds say, or one minute, into contact with the insulated plate of the condenser at any convenient point, such for example as the electrode of the electrometer, connected with the insulated pair of quadrants. The battery electrode is then removed, and the condenser and electrometer left to themselves.

The condenser has thus been charged to the potential of the battery, which will be indicated by the reading on the electrometer scale at the instant when the battery is removed. The deflection of the electrometer needle will

now fall, more or less slowly according to the insulation resistance of the condenser with its plates connected by the material being tested. Readings of the position of the spot of light on the electrometer scale, are taken at equal intervals of time, and recorded, and this is continued until the condenser has lost a considerable portion, say half, of its potential.

The resistance of the insulating material is easily calculated from the results in the following manner. Let V be the difference of potentials between the plates of the condenser at any instant, Q the charge of the condenser at that instant which may be taken as proportional to the deflection on the electrometer scale, and c_ϕ its capacity (p. 171). We have $Q = c_\phi V$, and therefore $\dfrac{dQ}{dt} = c_\phi \dfrac{dV}{dt}$.

But $- \dfrac{dQ}{dt}$ is the rate of *loss* of charge, that is, the current flowing from one plate to the other, and this is plainly equal by Ohm's law to $\dfrac{V}{R}$. Hence $- \dfrac{dQ}{dt} = \dfrac{V}{R}$ and therefore

$$c_\phi \frac{dV}{dt} + \frac{V}{R} = 0$$

Integrating we get,

$$\log V + \frac{t}{c_\phi R} = A, \quad . \quad . \quad . \quad . \quad (11)$$

where A is a constant. If V be the potential difference t seconds after it was V_0, we get by putting $t = 0$ in (11) $A = \log V_0$. Hence (11) becomes

$$\frac{t}{c_\phi R} = \log \frac{V_0}{V}$$

and

$$R = \frac{t}{c_p} \log \frac{V}{V_o} \, ^1 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (12)$$

If $V = \frac{1}{2} V_o$, we have $R = t \log \frac{1}{2}$.

If the condenser have a resistance so low as to add materially to the rate of discharge, an additional experiment must be made in the same way to determine the resistance of the condenser alone, with its plates connected only by its own dielectric. Let R_c denote the resistance of the condenser, determined by equation (5) from the results of the latter experiment, and R_i the resistance of the specimen, by equation (8) (p. 84) $\frac{1}{R} = \frac{1}{R_i} + \frac{1}{R_c}$, and therefore

$$R_i = \frac{R \, R_c}{R_c - R_i} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (13)$$

If c has been obtained in c.g.s. electrostatic units of capacity, it may be reduced to electromagnetic units by dividing by the second power of the number of electrostatic units of capacity which are equivalent to the electromagnetic unit, that is (p. 185) by 9×10^{20} nearly.

When an air condenser is used, its capacity can generally be obtained approximately by calculation from the dimensions and area of the plates. For example, if two parallel plates of metal, placed at a distance d apart, very small in comparison with any dimension of either surface, have a difference of potentials V, and there be no other conductor or electrified body near, there is very little electrification on the backs of the disks, and the charge per unit of surface on each of the opposite faces is numerically

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1 It is to be remembered that the logarithms to be here used are Naperian logarithms. The Naperian logarithm of any number is equal to the ordinary or Briggs' logarithm multiplied by 2·302585.

equal to $\dfrac{V}{4\pi d}$ at all points not near an edge. Hence
the charge on a portion of area A near the centre of
either plate is $\dfrac{A\,V}{4\pi d}$, and therefore (p. 171) the capacity
in electrostatic units of this portion of the plate is $\dfrac{A}{4\pi d}$.
Hence in the example below, we have, if we neglect the
effects of the non-uniformity of the electrical distribution
caused by the edges of the smaller disk, for the capacity
of the disk of area A the value $\dfrac{A}{4\pi d}$.

   In his attracted disk electrometers, Sir William Thom-
son has almost completely eliminated the effects of the
edges of the movable disk, by making it, when in its
proper position, form part of a plane surface extending
to a considerable distance all round in the plane of the
disk, and placing opposite to it a much larger disk. This
surrounding plane surface he has called the Guard Ring
or Guard Plate. The capacity of an air-condenser fitted
with a guard plate, which (to prevent electrification on
the back) forms part of a metal case surrounding the
back of the smaller disk, can therefore be very accurately
determined, and affords a convenient means of measuring
quantities of electricity.

   If c_* has been taken in absolute c.g.s. electromagnetic
units of capacity (p. 177), we obtain R from (12) in cms. per
second, which may be reduced to ohms by dividing by 10^9.

   When a condenser such as one of those used in sub-
marine telegraph work is used, the capacity of which is
known in microfarads (p. 182), then since a microfarad is
$\dfrac{1}{10^{15}}$ c.g.s. electromagnetic units of capacity, we have for

R in ohms the formula

$$R = 10^6 \frac{t}{c_p} \log \frac{V}{V_o} \quad . \quad . \quad . \quad (14)$$

The following are results actually obtained in tests of a specimen of insulating material made in the form of an ordinary telegraph insulator. An air condenser consisting of two horizontal brass disks, the distance of which apart could be regulated by means of a micrometer screw, was joined with the insulator, made into a small condenser with mercury inside and outside, as described above. The lower disk was of considerably greater diameter than the upper, which had a diameter of 12·54 cms., and the distance between them was adjusted to be 1 cm. The upper disk was connected to the insulated pair of quadrants, and the lower to the electrometer case. Calling A the area of the upper plate, and d the distance between them, we have, neglecting the effect of the edges of the upper disk, for the capacity of this condenser the value $\dfrac{A}{4 \pi d}$ in c.g.s. electrostatic units. Hence in the actual case $c_p = 9\cdot828$. The interior surface of the insulator, covered by the mercury, was so small, and the thickness of the material so great, that, even allowing the material to have a high specific inductive capacity, the capacity of the condenser which it formed was small in comparison with that of the air condenser. The experiment gave, when the condenser and insulator were joined as described, $V_o = 251$, $V_1 = 100$, $t = 5640$ seconds. Hence,

$$R = \frac{5640}{9\cdot828 \times 2\cdot303 \times \log_\epsilon \frac{251}{100}} = 623,$$

in seconds per centimetre (c. g. s. electrostatic units of

resistance). As the condenser was not insulating per-
fectly, a separate test was made for it alone, with the
results $V_o = 239$, $V_1 = 182$, $t = 6120$. Hence,

$$R_c = \frac{6120}{9 \cdot 828 \times 2 \cdot 303 \times \log_e \frac{239}{182}} = 2286,$$

and therefore by (13)

$$R_i = \frac{623 \times 2286}{2286 - 623} = 857,$$

in seconds per centimetre.

Multiplying this result by 9×10^{20} (the approximate
value of v^2, p. 185 below), to reduce to electromagnetic
units, we get for the resistance of the insulator $7712 \times
10^{20}$ cms. per second, or 771×10^{12} ohms.

We shall now consider very briefly the measurement of
the resistance of a battery. This term is not perfectly
definite in meaning, as there is reason to believe that the
resistance as well as the electromotive force of a battery
depends to some extent on the current flowing through
the battery, and further the resistance and the electro-
motive force, and possibly also the polarization of the
battery are affected by differences of temperature. But the
information which in practice we generally require from
the test, is really what available difference of potentials
can be obtained with a certain working resistance in the
external circuit. This could be obtained at once by con-
necting the terminals of the battery by this resistance,
and measuring the difference of potentials by means of a
quadrant electrometer or a potential galvanometer. If
we call this difference of potentials V, and the electro-
motive force of the battery when on open circuit E, then

putting R for the external resistance we may write

$$\frac{E}{R + r} = \frac{V}{R} = C \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where r is a quantity the definition of which is simply that it satisfies this equation. If the battery had the same electromotive force E, when generating the current C, as when on open circuit, then r would be the effective resistance of the battery; but, although this is not the case, we may without being led into error still speak of it as the resistance of the battery for the current C. In fact, the value of r, thus found for a particular value of R, does actually enable us to calculate from the known electromotive force for open circuit, with a moderate degree of approximation in the case of a constant battery, and also, but less surely, in the case of a secondary battery, what available difference of potentials will exist between the terminals of the battery when connected by other and somewhat widely differing values of R, and therefore also to find what arrangement of a battery it will be best to adopt in any given circumstances. So far as this practical result is concerned, the numerous methods which have been devised for the determination of the resistance of a battery before any sensible polarization (which requires time to develop) has been set up are, though interesting in themselves, of no practical value, and we shall not here describe any of them.

From equation (13) we have

$$r = \frac{E - V}{V} R \quad . \quad . \quad . \quad . \quad . \quad (16)$$

To determine r therefore we have simply to measure with a potential galvanometer the difference of potentials which exists between the terminals of the battery when on open circuit, or connected only by the galvanometer

coil, the resistance of which we suppose to be very great in comparison with r, and again to measure in the same way the difference of potentials when the terminals are connected by a resistance R, also small in comparison with that of the galvonometer.[1]

If the galvanometer scale be graduated so that readings are proportional to the tangents of the corresponding angles, we have, if D be the deflection in the first case, and D' the deflection in the second case, the equation

$$r = \frac{D - D'}{D'} R \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Instead of a potential galvanometer a quadrant electrometer may be employed if the battery is not too large, and the same formula applies when D and D' are taken proportional to the tangents of the angles through which the mirror is turned.

A resistance coil, which may be of German silver wire, constructed as described in p. 102, should be used for the resistance connecting the terminals, and if the current passing through it be considerable its resistance should be determined when the current is flowing. This may be done by including in its circuit a current-galvanometer, and determining the current C through the wire in amperes, when V is read off in volts on the potential instrument. The resistance of the wire with that of the current-galvanometer is in ohms $\frac{V}{C}$, and this is to be used as the value of R in equation (15).

If a galvanometer of high resistance be not available, an approximate test can be made by means of a sensitive

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[1] If the battery consist of a large number of cells, it may be divided into sections and so tested, or each cell may have its resistance measured separately.

galvanometer of low resistance. The battery and gal-
vanometer are joined in series with a resistance R, and
again with a resistance R'. Let D and D' be the de-
flections, which must have a difference comparable with
either. Then, supposing E and r to be the same in both
cases, and putting G for the resistance of the galvano-
meter, we have

$$D = m \, \frac{E}{R + G + r}, \quad D' = m \, \frac{E}{R' + G + r},$$

where m is a constant.

Therefore we find

$$r = \frac{D'R' - DR}{D - D'} - G. \quad . \quad . \quad . \quad (18)$$

CHAPTER X.

THE MEASUREMENT OF ENERGY IN ELECTRIC CIRCUITS.

WHEN a circuit in which a current of electricity is flowing contains a motor, or machine by which work is done in virtue of electromagnetic action, the whole electrical work done in the circuit consists, as was first shown by Joule, of two parts, work spent in heat in the generator and motor and in the conductors connecting them, and work done in moving the motor against external resistance. The total rate at which electrical energy is given out in the circuit is, as we have seen, $E\,C$ watts, where E is the total electromotive force in the circuit in volts, and C is the number of amperes of current flowing. The rate at which work is spent in heat is in watts, by Joule's law, $C^2\,R$, where R is the total resistance in circuit in ohms ; hence, if we call W the rate at which work is done in the motor,[1] we have,

$$E\,C = C^2\,R + W \quad . \quad . \quad . \quad (1)$$

We may write this equation in the form,

$$C = \frac{E - \dfrac{W}{C}}{R} \quad . \quad . \quad . \quad . \quad (2)$$

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[1] We considerer here a system in which C is constant, and neglect loss of energy due to local currents, &c., in the motor. For fuller information regarding motors and their action see a paper by Profs. Ayrton and Perry, *Proc. Soc. Tel. Eng.*, 1883, republished in the electrical journals.

which shows that the current flowing is equal to that which would flow in the circuit if, the resistance remaining the same, the motor were held at rest, and the electromotive force diminished by an amount equal to $\dfrac{W}{C}$. This is what is called the *back electromotive force* of the motor, and is due to the action of the motor in setting up when driven an electromotive force tending to send a current through the circuit in the opposite direction to that of the current by which the motor is driven. We shall denote the back electromotive force by E_1. Hence equation (2) becomes,

$$C = \frac{E - E_1}{R} \quad . \quad . \quad . \quad . \quad (3)$$

and the rate at which work is spent in driving the motor is $E_1 C$.

To determine E we have simply to measure with a potential galvanometer or voltmeter, the difference of potentials between the two terminals of the generator. Calling this V, and R_1 the effective resistance of the generator, we have plainly,

$$E = V + C R_1 \quad . \quad . \quad . \quad . \quad (4)$$

Again, since C and also the total resistance R in the circuit can be found by measurement, we find by (3)

$$E_1 = E - C R \quad . \quad . \quad . \quad . \quad (5)$$

where all the quantities on the right-hand side are known.

The ratio of $E_1 C$, the electrical energy spent per unit of time in the circuit otherwise than in heating the conductors, to the whole electrical energy $E C$ spent in the circuit per unit of time, that is the ratio of E_1 to E, we may call the electrical efficiency of the arrangement. Denoting

this efficiency by e, we find, by equation (4),

$$e = \frac{E_1}{E} = 1 - \frac{C R}{E} = 1 - \frac{E - E_1}{E}.$$

Hence the smaller C is made, that is, the slower the energy is given out, the value of the efficiency of the arrangement is the more nearly equal to *unity*, the value of the efficiency of an arrangement in which the energy in the motor done against external resistance is exactly equal to the whole electrical energy given out in the circuit.

When however energy is spent at the maximum rate in working the motor, $E_1 C$ has its greatest value. But by (4)

$$E_1 C = E C - C^2 R = W.$$

This equation may be written,

$$C^2 R - E C + W = 0,$$

a quadratic equation of which the solution is,

$$C = \frac{E \pm \sqrt{E^2 - 4 R W}}{2 R}$$

Now in order that these values of C may be *real*, $4 R W$ cannot be greater than E^2. Hence the greatest value, W can have is $\frac{E^2}{4 R}$. When W has this maximum value, C is equal to $\frac{E}{2 R}$, and therefore E_1 equal to $\frac{E}{2}$. Hence the electrical efficiency is ½. It is to be very carefully observed that although in this case the arrangement is that of *greatest electrical activity* it is *not that of greatest electrical efficiency*, as it has only about one-half the efficiency of one in which energy is given out at a very slow rate. The case is exactly analogous to that referred to in p. 81, of a battery arranged so as to give maximum current through a given external resistance.

All that has been stated above is applicable to the case of a motor fed by any kind of generator whatever. The generator employed however is generally some form of dynamo- or magneto-electric machine driven by an external motor, such as a steam- or gas-engine or a water-wheel, and a few of the results obtained below apply only to such cases, which will be indicated as they occur.

When the generator and motor are exactly similar machines, and the same current passes through both, the ratio of the forces E_1 and E will be that of $n\,A\,f(C)$ to $n'\,A\,f(C)$; where n and n' are the speeds of the machines, A a constant depending on the form and disposition of the magnets, and $f(C)$ a function of the current. Hence in this case the efficiency is measured simply by the ratio of the speed of the motor to that of the dynamo. The more nearly therefore the speed of the motor approaches to that of the generator, the greater is the efficiency.

In general, the higher the speed at which the motor is run, the greater is the electrical efficiency of any arrangement, for it is obvious that the higher the speed the more nearly does E_1 approach to E, and therefore the value of $\dfrac{E_1}{E}$, the measure of the efficiency, to unity.

For a constant difference $E - E_1$, the ratio of the energy spent in heating the conductors by the current to the whole energy expended in the circuit, may be reduced by increasing the total electromotive force E of the circuit. The energy spent in heat is $C^2 R$, or, $\dfrac{(E - E_1)^2}{R}$, and the ratio of this to $E\,C$ is $\dfrac{C\,R}{E}$. But $C\,R$ is equal to the constant difference $E - E_1$, hence the ratio is $\dfrac{E - E_1}{E}$,

and this becomes smaller as E is increased. A greater efficiency is therefore obtained by using high potentials than by using low potentials. Hence a greater electrical efficiency is realised, with a given magneto- or dynamo-electric machine used as generator and a given motor, when both generator and motor are run at higher speeds. Consequently the generator should be run as fast as possible, and the motor loaded lightly, or the speed with which the working resistance is overcome reduced by gearing between it and the motor.

When high potentials are obtained by the use of machines wound with fine wire, or by using as generator a battery of a large number of cells joined in series to drive a high potential motor, the gain of electromotive force is accompanied by an increase of resistance in the circuit. But if we suppose the speed of the motor to be so regulated that the difference between the total electro-motive force in the circuit and the back electromotive force of the motor remains the same in the different cases, it is easy to show that the electrical efficiency of the arrangement is greater for high electromotive forces than for low. If, as supposed, $E - E_1$ remains constant, while E is changed to $n E$, we have for the total activity of the motor $n E C - (E - E_1) C$. Dividing this by $n E C$ we get for the electrical efficiency,

$$e = \frac{n - 1}{n} + \frac{1}{n}\frac{E_1}{E} \quad . \quad . \quad . \quad . \quad (7)$$

As n is made greater and greater, the first term on the right becomes more and more nearly equal to unity, and the last term to zero. Hence, on the supposition made, the efficiency is increased by increasing the working electromotive forces. Taking as a particular case $n = 2$, we see that the efficiency is $\frac{1}{2}$ together with one-half of the

former efficiency; if $n = 4$, the efficiency is $\frac{3}{4}$ together with one-fourth of the former efficiency, and so on for other values of n. This result holds for any case whatever in which the condition that $E - E_1$ should remain constant is fulfilled; and hence it is independent of any change that may have been made in the resistances of the generator or motor in order to obtain the higher electromotive force $n E$. For example, it is plain that no sensible change in the actual rate of loss by heating of the conductors by the current will be produced by increasing the resistances of the generator and motor, if these be very small in comparison with the remainder of the resistance in circuit; as, since $E - E_1$ remains constant and the resistance is practically the same as before, the current strength will not be perceptibly altered. The ratio, however, of the activity wasted in heating to the total activity will be only $\frac{1}{n}$th of what it was before. In the opposite extreme case, in which the generator and motor have practically all the resistance in circuit, the current, $C \left(= \dfrac{E - E_1}{R} \right)$, is diminished in the ratio in which the resistance is increased; and the actual rate of loss by heat according to Joule's law, $\dfrac{(E - E_1)^2}{R}$, is diminished in the same ratio, so that, as in the former case, its ratio to the total activity $n E C$ is $\frac{1}{n}$th of what it was for the electromotive force E. We see, therefore, that here also the efficiency must be the same in both cases.

We have called $\dfrac{E_1}{E}$ the *electrical efficiency of the arrangement*, but this is not to be confounded with the

efficiency of the motor itself. The activity $E_1 C$ includes the wasted activity or rate at which work is done against frictional resistances in the motor itself, and in the gearing which connects it with its load, as well as the useful activity or rate at which it performs useful work. Hence, although the electrical efficiency of the arrangement be very great, only a small amount comparatively of the energy given to the motor may be usefully expended, and *vice versâ;* and we define therefore the efficiency of a motor at any given speed as the ratio of the useful activity to the whole activity, taking as the latter the total rate at which electrical energy is expended in the motor; that is, $E_1 C + C^2 R_1$, or, which is the same, $V C$, where V is the difference of potentials between the terminals of the motor. Accordingly, if A be the useful activity, we have for the efficiency of the motor the ratio $\dfrac{A}{V C}$.

To determine this ratio in any particular case the motor is run at the required speed, V is measured with a potential galvanometer, and C with a current galvanometer, and their product taken, or $V C$ is determined with some form of electrical activity-meter, while A is determined by means of a suitable ergometer. A very convenient and accurate friction ergometer may be formed by passing a cord once completely round the pulley of the motor, and hanging a weight on the downward end, while the other is made to pull on a spiral spring fixed at its upper end, and provided with an index to show its extension. The weight is adjusted so that the motor runs at the required speed, while wasting all its work in overcoming the friction of the cord, and the extension of the spring is noted, and the corresponding pull found in the same units of force as those used in estimating the

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downward pull due to the weight. Let the weight used in any experiment be taken in grammes, and be denoted by w, and let w' be the number of grammes required to stretch the spring by gravity to the same amount, then the total force overcome is in dynes $(w - w')\, g$, where g is the acceleration, in centimetres per second per second, produced by gravity at the place of experiment (at Glasgow $g = 981\cdot4$ nearly). If n be the number of revolutions per second, and c the circumference in cms. of the pulley at the part touched by the rope, the velocity with which this force is overcome is $n\,c$, and therefore the activity in ergs per second is $n\,c\,(w - w')\, g$. If A is reckoned in watts, we have the equation,

$$A = \frac{1}{10^7}\, n\, c\, (w - w')\, g.$$

If $w - w'$ be taken in pounds, and c in feet, and n be the number of revolutions per *minute*, the activity in horse-power is given by

$$A = \frac{1}{33000}\, n\, c\, (w - w')$$

and in watts approximately by

$$A = \cdot0226\, n\, c\, (w - w').$$

We have now considered cases in which electrical energy is transformed into mechanical work by means of motors working by electromagnetic action, and have seen that the whole electrical activity $E\,C$ in the circuit is equal to the useful activity of the motor together with the unavailable part spent in heating the conductors in circuit, and in overcoming the frictional resistances opposing the motion of the motor. Part of the electrical energy developed by a generator may however be spent in effecting chemical decompositions in electrolytic cells placed in the circuit, as, for example, in charging a secondary battery or

"accumulator." Each cell in which electrolytic action takes place, so that the result is chemical separation at the plates of the constituents of the solution acted on, opposes a counter electromotive force to that causing the current to flow, and the work done in each cell over and above that spent in heat, according to Joule's law (p. 47) is equal to the product of this counter electromotive force into the strength of the current. In most cases the counter electromotive force exceeds the electromotive force required to effect the chemical decompositions, and the energy corresponding to the difference of electromotive force appears in the form of what has been called *local heat* in the electrolytic cells.

In the case of a secondary battery charged by the current from an electrical generator, which is the only case we shall here consider, the activity spent in the battery while it is being charged is equal to the product of the difference of potentials existing between the terminals of the battery while the current is flowing, multiplied by the strength of the current. Let V be this difference of potentials in volts, and C the current strength in amperes, then VC joules is the whole work per unit of time spent in the battery. The whole activity spent in the circuit is EC, or $VC + C^2 R$, where E is the total electromotive force of the generator, and R is the resistance of the generator and the wires connecting it with the secondary. Again if E_1 volts be the electromotive force of the secondary battery, which may be measured by removing the charging battery for an instant and applying a potential galvanometer to the terminals of the secondary, the activity actually spent in charging the battery may be taken as $E_1 C$. Hence the ratio of the activity spent in charging the battery to the

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whole activity in the circuit is $\dfrac{E_1}{V + RC}$ or $\dfrac{E_1}{E}$, and the activity wasted in heating the conductors in circuit is $(E - E_1)\,C$. This ratio $\dfrac{E_1}{E}$ is the same as that found above in the case of a generator and a motor, and may be called as before the electrical efficiency of the arrangement.

Hence, in order that as nearly as possible the whole electrical energy given out in the circuit may be spent in charging the battery, as many cells should be placed in circuit as suffice to nearly balance the electromotive force E of the generator, that is, the charging should be made to proceed as slowly as possible. In practice, however, a very slow rate of charging is not economical, as the work spent in driving the generator, if a dynamo- or magneto-electric machine, against frictional resistances would be greater than the useful work done in the circuit; and if the speed of the generator slackened for a little the battery would tend to discharge through it.

As in the case of the motor (p. 127), the electrical efficiency of the arrangement can be increased by increasing E and E_1, so that $E - E_1$ is maintained constant. E may, in the present case, be increased by running the generator faster, or by using a machine adapted to give higher potentials. As before, if E be increased to nE, while E_1 is changed to E_1' so that $n\,E - E_1' = E - E_1$, the electrical efficiency becomes $\dfrac{n - 1}{n} + \dfrac{1}{n}\dfrac{E_1}{E}$.

The electromotive force of a Faure cell is about 2·2 volts when fully charged, but is considerably less when nearly discharged. When the cell is placed in the charging circuit, the counter electromotive force which it opposes rises quickly to a little less than this value, and thereafter

gradually increases, while the charging current falls in strength. In order to measure, therefore, the whole energy spent in charging a secondary battery, we must either use some form of integrating energy-meter which gives accurate results, or measure, at short intervals of time, V with a potential galvanometer, and C with a current galvanometer placed permanently in the circuit. After the battery has been charged, the total number of joules spent is obtained by multiplying each value of VC by the number of seconds between the instant at which the corresponding readings were taken and that at which the next pair of readings were taken, and adding all the results. If, as generally will be the case, the readings vary nearly uniformly over the time occupied in obtaining several pairs, the mean product for these may be multiplied by the total number of seconds. The integral work in joules having been thus estimated, the efficiency of the battery may be obtained by finding in the same manner the total number of joules given out in the external working circuit while the battery is discharging. The ratio of the useful work thus obtained to the whole work spent in charging is the efficiency of the battery. In discharging in an electric light circuit, the greatest economy is obtained when the resistance of the working part of the circuit is very great in comparison with that of the battery and main conductors. Neglecting the latter part of the resistance, we see that, if a large number of lamps are arranged in multiple arc, a large number of cells should also be joined in multiple arc, so that, while the requisite potential is obtained, the resistance of the battery is still small in comparison with that of the external circuit.

As regards the measurement of energy spent in electric light circuits, in which continuous currents are flowing,

we have already sufficiently indicated above (Chap. VII.)
how this may be done.   To find the activity, or work spent
per unit of time, in any part of a circuit, we have only to
find the difference of potentials, V, in volts between its
extremities with a potential galvanometer, and the current,
C, in amperes flowing through it with a current instrument.
If the activity be constant, we have simply to multiply VC
by the number of seconds in any interval of time, to find the
number of joules spent in that time in the part of the circuit
in question.   If the activity is variable, the whole energy
spent in any time may be estimated by finding VC at short
intervals of time, and calculating the integral as explained
above (p. 133).

So far we have been considering only measurements
made in the circuits of batteries or of continuous current
generators.   Alternating machines in which the direction
of the current is reversed two or three hundred times a
second are, however, frequently employed, especially in
electric light circuits, and it is necessary therefore to
consider the methods of electrical measurement available
in such cases.   We shall consider briefly, first, a class of
instruments some of which can be used in a circuit of
either kind, and we shall deal in the first place with their
application in continuous current circuits.

These are instruments the fundamental principle of
which is the mutual electro-magnetic action between two
circuits, to be calculated conveniently in most cases in
which this can be done, by replacing each circuit according
to Ampère's theory, by an equivalent magnetic shell (p. 25),
and considering the mutual action of the systems. One con-
venient form for absolute measurements is that known as
Weber's Electro-dynamometer.[1]   Such an instrument may

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[1] See Maxwell's *Electricity and Magnetism*, vol. ii. p. 330.

be constructed by replacing the needle of the standard tangent galvanometer (Chap. IV.) with a coil of radius small in comparison with that of the galvanometer coil, and suspended by a torsion wire or bifilar, so as to hang in equilibrium when no current is passing through it, with its plane at right angles to that of the large coil. Hence when a current C is made to flow through the suspended coil, and a current C' through the suspended coil, a couple is exerted on the latter, tending to set it with its plane parallel to the large coil, and this tendency is resisted by the action of the torsion or bifilar suspension, so that there is equilibrium for a deflection θ, the magnitude of which plainly depends on the product $C\,C'$ of the two current strengths. To avoid disturbance from the action of the local horizontal magnetic force, the large coil may be placed parallel to the direction of that force, and the small coil brought back when deflected by the current to the initial position at right angles to the large coil, by turning the upper end of the suspension wire or wires through a measured angle.

By Ampère's theorem (p. 24) the suspended coil is equivalent to a small needle of moment $n\,A\,C'$, where n is the number of turns of wire in the coil, A their mean area. Hence if N be the number of turns in the large coil, r its mean radius, we have as in p. 28 for the electromagnetic couple on the suspended coil, when brought back to the initial position, the value $\dfrac{2\,\pi}{r}\,N\,C\,.\,n\,A\,C'$ or

$\dfrac{2\,\pi}{r}\,n\,N\,A\,C\,C'$. This couple is balanced by the opposite couple given by the suspension, the magnitude of which for all angles within a certain range is supposed known

from experiment. Calling this latter couple L, we have

$$C\,C' = \frac{L\,r}{2\,\pi\,n\,N\,A}.$$

If the two coils be joined in series so that the same current flows through both, we have $C = C'$, and therefore

$$C^2 = \frac{L\,r}{2\,\pi\,n\,N\,A}.$$

With such an instrument therefore an absolute measurement of a current can be made without its being necessary first to determine H. In practical work the instruments on this principle usually employed are such as require to have their constants determined by comparison with standard instruments, such as a standard tangent galvanometer, or a standard dynamometer, and we shall not here enter into details regarding them. We may mention, however, Siemens' Electro-dynamometer, in which a suspended coil is acted on by a fixed coil, and the strength of the current deduced, by means of a table of values for different angles, from the torsion which must be given to a spiral spring to bring the coil back to the zero position; and the practical current-weighers invented by Sir William Thomson, by which currents, or potentials, or activities, are measured by weighing the forces on a coil, or coils, carried by a frame or beam turning round knife-edges, exerted by fixed coils properly arranged.

When an instrument on this principle is arranged for use as an activity-meter, one set of coils, the fixed or the movable, is made of thick wire so as to carry the whole current in the circuit, while the other is made of high resistance and is connected to the two ends of the part of the circuit in which the electrical activity is to be measured. In this case the force or couple required to restore the movable coils to the zero position is proportional to the

product VC of the difference of potentials and current, that is to the activity, for that part of the circuit; and if the instrument has been properly graduated this can be at once read off in watts, or in any other chosen units of activity. Instruments of this kind have been made by Professors Ayrton and Perry, Sir William Thomson, and Sir William Siemens.

We shall now consider the measurement of currents and potentials, and therefore also of electrical energy in the circuits of alternating machines. In all such machines the march of the current in each complete alternation may be stated roughly as a rise from zero to maximum in one direction, then a diminution to zero, then a change of sign and a rise to maximum in the opposite direction, followed by a diminution again to zero. The law according to which these changes take place is more or less complex in the various cases, and the complete mathematical representation of the current strength at any time would require an application of Fourier's method of representing any arbitrary periodic function, by means of an infinite series of simple harmonic terms of the form $a_n \sin\left(n\frac{\pi}{\tau}t - \epsilon_n\right)$ where τ is the half period of a complete alternation, a_n and ϵ_n constants and n any integer. It has been found experimentally by M. Joubert that the variation of electromotive force in a Siemens' alternating machine can be expressed by the single harmonic term $\frac{E_0}{\tau} \sin \frac{\pi}{\tau} t$, where we reckon t from the instant at which the electromotive force was zero when changing from the direction reckoned as negative to that reckoned as positive. There is good reason to believe that this law holds approximately for the majority

of alternating machines, and we shall assume its truth in
what follows.   The current strength is affected by the
action of self-induction (p. 92) to a greater or less extent
in all such machines independently of the disposition of
the external circuit, especially if the revolving armature
contains iron ; but, as shown below, it follows, with a
difference in phase, the same law as does the electro-
motive force.   The effect of mutual induction due to
variations in the field magnets produced by the rotating
armature has also in a rigorous theory to be taken into
account, but this effect, where experiments have been
made to detect it, has been found to be slight, and we
shall therefore neglect it.

Writing then C for the current, at a time t, less than τ,
reckoned from the instant at which the current was zero,
we have

$$C = \frac{a}{\tau} \sin \frac{\pi}{\tau} t. \quad . \quad . \quad . \quad . \quad (11)$$

The whole quantity of electricity generated in a half
period τ is therefore

$$\int_o^t C \, dt = \frac{a}{\tau} \int_o^\tau \sin \frac{\pi}{\tau} \, dt = \frac{2 \, a}{\pi}. \quad . \quad . \quad (12)$$

Hence if C_m denote the mean current in that time, we have

$$C_m = \frac{2 \, a}{\pi \, \tau} \quad . \quad . \quad . \quad . \quad (13)$$

Now if an electro-dynamometer be placed in the circuit
so that the same current passes through both its fixed and
movable coils, the current in both will be reversed at the
same instant, and their mutual action will be the same for
the same current strength, and will be proportional to C^2
that is to $\frac{a^2}{\tau^2} \sin^2 \frac{\pi}{\tau} t$.   If the period of the alternation be
small in comparison with the period of free oscillation of
the movable coil system of the dynamometer, the mutual

action of the fixed and movable coils will be the same as if a continuous current C' given by the equation

$$C'^2 = \frac{1}{\tau}\int_o^\tau C^2\, dt = \frac{a^2}{\tau^3}\int_o^\tau \sin^2\frac{\pi}{\tau} t\, dt \quad . \quad (14)$$

were kept flowing through them. But by integration

$$C'^2 = \frac{a^2}{2\,\tau^2} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and substituting from (13) in this equation, we get

$$C_m = \frac{2\,\sqrt{2}}{\pi}\, C' = \text{·}900\, C'. \quad . \quad . \quad (16)$$

In order therefore to find the actual mean current strength in the circuit of an alternating machine from the value of C' given by a current dynamometer we must multiply the latter by ·9. The product, if C' has been taken in amperes, multiplied by the number of seconds in any interval of time during which the machine has been working uniformly on the same circuit, will give the number of coulombs of electricity which has flowed through the circuit in that time.

The measurement of potentials is however attended with more difficulty on account of the effect of the self-induction of any electromagnetic instrument which can be applied to the circuit for this purpose. The following method of employing Sir William Thomson's quadrant electrometer for this purpose has been used by M. Joubert.[1] The needle of the instrument is left uncharged, and the charging rod connected with it and used as a third electrode. If then we suppose the needle connected to one point in the circuit at which the potential is V, one pair of quadrants at a point at which the potential is V_1, and the other pair at a third

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[1] *Comptes Rendus*, July, 1880. *Annales de Chimie et de Physique*, May, 1883.

point where the potential is V_2, then if D be the deflection of the spot of light corresponding to the angle (supposed small) through which the needle has been turned against the bifilar suspension, we have

$$D = k\left(V_1 - V_2\right)\left(V_1 - \frac{V_1 + V_2}{2}\right) \quad . \quad (17)$$

where k is a constant.[1]   If the needle be connected

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[1] This formula holds for any symmetrical electrometer, which in its elementary form consists of three conductors maintained at different potentials, and fulfilling the following conditions:—One of the conductors (A) (in the quadrant electrometer the needle) is symmetrically placed with reference to the other two (B and C) and is so formed that one of its two ends or bounding edges is well under cover of B, and the other end or edge under cover of C, so that the electrical distribution near each end or edge is uninfluenced except by the nearer conductor. Let the potentials of A, B, C be respectively V, V_1, V_2; and let A be slightly displaced from B towards C. This displacement may be angular or linear, according to the arrangement adopted ; in the quadrant electrometer it is measured by the angle through which the needle is turned. Let θ denote the displacement and c_ϕ the electrostatic capacity of A per unit of θ at places not near the ends or bounding edge of A, and well under cover of B and C. Then the quantity of electricity lost by A in consequence of its displacement relatively to B is $c_\phi \theta (V - V_1)$, and the quantity lost by B is $c_\phi \theta (V_1 - V)$. Similarly the quantities gained by A and C in consequence of the motion of A towards C are respectively $c_\phi \theta (V - V_2)$ and $c_\phi \theta (V_2 - V)$. Multiplying the first and second of these quantities by V and V_1 respectively, the third and fourth similarly by V and V_2, subtracting the sum of the first two products from the sum of the second two, and dividing by 2, we get for the work done by electrical forces in the displacement the value $c_\phi \theta (V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right)$. But this must be equal to the average couple multiplied into the displacement if the latter is angular, or the average force into the displacement if the latter is linear. We have therefore, denoting the force or couple by F,

$$F = c_\phi (V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right).$$

In an arrangement of this kind when the displacement is small the couple or force acting on A is nearly the same over the whole displacement, and thus is nearly equal to the equilibrating couple or force due to the torsion wire, or bifilar, or other arrangement producing equilibrium. But for small displacements this will generally be proportional to the displacement, and therefore also to the deflection D on the scale of the instrument, and thus we have

$$D = m\theta = k(V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right)$$

where m and k are constants.
  When V is great in comparison with V_1 and V_2 this reduces to $\theta = k'(V_1 - V_2)$ the equation employed when, as in the ordinary use of the quadrant electrometer, the needle is kept charged to a constant high potential.

to the pair of quadrants whose potential is V_1, we have

$$D = \frac{k}{2}(V_1 - V_2)^2 \quad . \quad . \quad . \quad (18)$$

The constant k may be determined by applying the electrometer as thus arranged to the terminals of a Daniell's battery, the electromotive force of which is known, or by measuring a constant difference of potentials simultaneously with the electrometer and a potential galvanometer.

If the quadrants be connected to any two points in the circuit of a machine in which the period of alternation is short in comparison with the free period of the needle, the couple acting on the needle will be at each instant proportional to the second power of the difference $V_1 - V_2$ of potentials existing between these two points at that instant, and this of course is independent of the sign of the difference. Also as in the similar case of the dynamometer above, the deflection of the needle will be the same as if the second power $(V_1 - V_2)^2$ of the difference of potentials had a mean value $\frac{1}{\tau}\int_o^\tau (V_1 - V_2)^2 \, dt$, and if we denote the square root of this mean value by V', and the actual mean difference of potentials by V_m, then since the difference of potentials follows the same law of variation as the current, we get also

$$V_m = \cdot 900 \, V' \quad . \quad . \quad . \quad . \quad . \quad (19)$$

If we know the resistance in the part of the external circuit between the points at which the electrometer electrodes are applied, then calling this resistance R, and supposing that this part of the circuit contains no motor or other arrangement giving a back electromotive force,

and that its self-induction is zero or negligible in comparison with R, we have for the mean value of the current $\dfrac{V_m}{R}$, and thus by means of an electrometer alone we can measure not only the difference of potentials between the ends of, but also the current in, that portion of the circuit.

Denoting by A_m the mean value of the electrical activity in this part of the circuit, still supposing the self-induction of this part to be negligible, we have plainly

$$A_m = \frac{1}{R\,\tau} \int_o^\tau (V_1 - V_2)^2\, dt = \frac{V'^2}{R} \quad . \quad . \quad (20)$$

In the same way, since the value of the electrical activity at any instant is $C^2 R$, we have from the results of experiments made by an electro-dynamometer,

$$A_m = \frac{R}{\tau} \int_o^\tau C^2\, dt = C'^2 R \quad . \quad . \quad . \quad (21)$$

From these two results we get

$$A_m = V' C' \quad . \quad . \quad . \quad . \quad (22)$$

that is, the mean value of the electrical activity is equal to the product of the square root of the mean square of the difference of potentials, by the square root of the mean square of the current strength. It can therefore be determined by means of an electrometer and an electro-dynamometer of negligible self-induction without its being necessary to know the resistance.

We shall now consider the case in which the self-induction cannot be neglected. Let R be the total resistance in the circuit, C the current flowing in it at the time t, E the total electromotive force of the machine, and L the coefficient of self-induction for the whole circuit, that is, the number which multiplied

into $\dfrac{dC}{dt}$ gives the electromotive force opposing the increase or diminution of the current. We have

$$R C = E - L \dfrac{d C}{d t} \quad . \quad . \quad . \quad . \quad (23)$$

But by the law which we have assumed for the machine,

$$E = \dfrac{E_0}{\tau} \sin \dfrac{\pi}{\tau} t \quad . \quad . \quad . \quad . \quad (24)$$

where E_0 is the maximum which E would have if the speed were such as to give only one complete alternation in two seconds, and t is reckoned from the instant at which E is zero. Substituting in (23) we get the linear differential equation

$$L \dfrac{d C}{d t} + R C = \dfrac{E_0}{\tau} \sin \dfrac{\pi}{\tau} t \quad . \quad . \quad . \quad (25)$$

which integrated becomes

$$C = A e^{-\frac{R}{L} t} + \dfrac{E_0}{\sqrt{R^2 \tau^2 + \pi^2 L^2}} \sin \left(\dfrac{\pi}{\tau} t - \epsilon \right) \quad . \quad (26)$$

where

$$\sin \epsilon = \dfrac{\pi L}{\sqrt{R^2 \tau^2 + \pi^2 L^2}}, \quad \cos \epsilon = \dfrac{R \tau}{\sqrt{R^2 \tau^2 + \pi^2 L^2}}. \quad (27)$$

The term $A e^{-\frac{R}{L} t}$ is only important immediately after the circuit is closed, and will therefore be neglected.

We may remark that if L were equal to zero (26) would reduce to $C = \dfrac{E_0}{\tau R} \sin \dfrac{\pi}{\tau} t$ which corresponds to (11) above.

From (26) we get for the mean current

$$C_m = \dfrac{E_0}{\tau \sqrt{R^2 \tau^2 + \pi^2 L^2}} \int_{\frac{\epsilon \tau}{\pi}}^{\frac{\epsilon \tau}{\pi} + t} \sin \left(\dfrac{\pi}{\tau} t - \epsilon \right) dt = \dfrac{2 E_0}{\pi \sqrt{R^2 \tau^2 + \pi^2 L^2}} \quad (28).$$

Also for the mean square of the current strength as given directly by an electro-dynamometer we have by (25) the equation

$$C'^2 = \frac{E_0^2}{\tau (R^2\tau^2 + \pi^2 L^2)} \int_{\frac{\epsilon\tau}{\pi}}^{\frac{\epsilon\tau}{\pi}+\tau} \sin^2 \left(\frac{\pi}{\tau}t - t\right) dt$$

$$= \frac{1}{2} \frac{E_0^2}{R^2\tau^2 + \pi^2 L^2} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (29)$$

and we have therefore as before, the relation

$$C_m = \cdot 900 \; C'.$$

From (25) and (26) we see that the effect of self-induction is to diminish every value of the current in the ratio of $\dfrac{E_0}{\sqrt{R^2\tau^2 + \pi^2 L^2}}$ to $\dfrac{E_0}{R\tau}$, and to produce a retardation of phase which measured in time is $\dfrac{\epsilon\tau}{\pi}$ seconds ; that is, the current in following the law of sines passes through any value $\dfrac{\epsilon\tau}{\pi}$ seconds after it would have passed through the corresponding value if there had been no self-induction. It is plain also that, for any finite resistance R, by diminishing τ, that is, by increasing the speed of the machine, the current can be made to approach the limiting value

$$C = \frac{E_0}{\pi L} \sin \left(\frac{\pi}{\tau}t - \frac{\pi}{2}\right) \quad \cdot \quad \cdot \quad \cdot \quad (30)$$

which is independent of the resistance, and has a retardation of phase of $\dfrac{\tau}{2}$ seconds, that is of a quarter period of a complete alternation. Hence integrating over a half period

from zero current to zero current again, and dividing by τ we get for the maximum mean current

$$C_m = \frac{2\,E_0}{\pi^2\,L} \quad . \quad . \quad . \quad . \quad . \quad (31)$$

To find the mean value A_m of the total electrical activity in the circuit, we have by (24), (26), and (27)

$$A_m = \frac{1}{\tau}\int_0^\tau E\,C\,dt = \frac{E_0^2}{\tau^2\,\sqrt{R^2\tau^2 + \pi^2 L^2}}\int_0^\tau \sin\left(\frac{\pi}{\tau}t - \epsilon\right)\sin\frac{\pi}{\tau}t\,dt$$

$$= \frac{1}{2}\,\frac{E_0^2\,R}{R^2\,\tau^2 + \pi^2\,L^2} \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Hence by (29)

$$A_m = C'^2\,R \quad . \quad . \quad . \quad . \quad (33)$$

that is, the mean value of the total electrical activity is equal to the mean square of the current strength multiplied by the total resistance in circuit.

It may be shown, from (32), by the method used in pp. 80, 81, above, that the total activity in the circuit is greatest when $R = \dfrac{\pi\,L}{\tau}$, that is, *for a given speed and a given value of L,* the activity is a maximum when $R = \dfrac{\pi\,L}{\tau}$. It must be observed however that *for a given resistance R* the activity is greater the smaller the value of τ, that is, the greater the speed. When R has the value $\dfrac{\pi\,L}{\tau}$ we have, by (27), $\epsilon = \dfrac{\pi}{2}$; that is, the retardation of phase is then one eighth of the whole period.[1]

We may apply the formulas here given for the whole circuit to a part, taking for E the impressed electromotive force on the part of the circuit considered, and for R and

[1] The conclusions as to maximum work and retardation of phase, as well as most of the theoretical results stated above as to the action of alternating machines, were first we believe given by M. Joubert, *Comptes Rendus,* 1880. It has been assumed in the above investigation, that there are no masses of metal, in which local currents can be generated, moving in the field, and the conclusions are applicable only in that case.

L the proper values for that part only. We find that the effect of self-induction is virtually to increase the resistance from R to $\frac{1}{\tau}\sqrt{R^2\tau^2 + \pi^2 L^2}$, and to produce a difference of phase between the current and E given also by (27).

If the circuit divide into two parts, each forming a derived circuit on the other, and $L_1, L_2, R_1, R_2, C_1, C_2$ be the coefficients of self-induction, the resistances, and the maximum currents in the two parts, C the maximum total current in the circuit, and ϵ_1, ϵ_2 the differences of phase between C and C_1, C_2 respectively, a similar analysis shows that if there be no mutual induction

$$\frac{C_1^2}{R^2\tau^2 + \pi^2 L_2^2} = \frac{C_2^2}{R_1^2\tau^2 + \pi^2 L_1^2} = \frac{C^2}{(R_1 + R_2)^2\tau^2 + \pi^2(L_1 + L_2)^2}$$

$$\tan \epsilon_1 = \frac{\pi\,\tau\,(L_1 R_2 - L_2 R_1)}{(R_1 + R_2)\,R_2\,\tau^2 + \pi^2\,(L_1 + L_2)\,L_2}.$$

with a similar formula for ϵ_2. Hence the difficulty in using an electromagnetic instrument to measure currents or potentials directly in a derived circuit.

As we have seen above, the proper mean value of the current, and of the difference of potentials, and therefore also of the activity, can be found for any part of a circuit in the case of negligible self-induction, either by means of an electro-dynamometer, or by means of an electrometer, when the resistance of the part of the circuit is known. When the resistance is unknown or uncertain, as for example in the case of incandescence lamps, the current and difference of potentials may be measured for the lamp circuit in the following manner. A coil of german silver wire, having a resistance considerably greater than that of the lamps as arranged, constructed as described above (p. 102), so as to have no self-induction, is connected in series with a current-meter between the terminals of the machine so as to be a shunt on the lamps.

The lamps are brought to their normal brilliancy, and the mean square C'^2 of the current through the german silver wire measured. If R be the resistance of this wire, including, if appreciable, the resistances of the current-meter and its connections, and R be great in comparison with the coefficient of self-induction of the current-meter divided by τ, we have for the mean square V'^2 of the difference of potentials between the terminals of the lamp system, the value $C'^2 R^2$. The current-meter is now employed to measure the whole current flowing to the lamps while their brilliancy is kept the same. Denoting the mean square of this current by $C_1'^2$, we have for the value A_m of the mean activity spent in the lamp system, the equation

$$A_m = V' C' = C' C_1' R \quad . \quad . \quad (34)$$

An electrometer may be used in the following manner to give the mean square of the current, and of the difference of potentials for any part of a circuit whether containing motors, or arc lamps, or any arrangement with or without counter-electromotive force or self-induction. A coil of thick german silver wire (or to prevent sensible heating a set of two or more equal coils arranged in multiple arc) having no self-induction is included in the part of the circuit considered, so that the current to be measured also flows through the wire. The mean square of the difference of potentials between the ends of this resistance is measured as described above (p. 141) by connecting one pair of quadrants of the electrometer to one end and the needle and the other pair of quadrants to the other end, and the mean square C'^2 of the current found by dividing by the square of the resistance of the wire. The mean square V'^2 of the difference of potentials between the terminals of the part of the circuit considered, is then found in the same manner. The product is not generally to be taken as

the mean square of the activity in the part of the circuit considered, for it is evident that in this case what is obtained is the value of $\frac{1}{\tau^2}\int_0^\tau V^2\,dt \times \int C^2\,dt$, where V and C are the difference of potentials and the current at any instant. The square root $\frac{1}{\tau}\sqrt{\int_0^\tau V^2\,dt \times \int_0^\tau C^2\,dt}$ of this quantity is not generally the same thing as $\frac{1}{\tau}\int_0^\tau VC\,dt$ the true mean value of the activity. This is, however, given directly by the following method.[1]

Let the two ends of the resistance coil of zero self-induction and known resistance R be called A and B, and let the extremities of the portion of the circuit for which the measurements are to be made be called C and D. One of the pairs of quadrants is connected to A, the other pair to B, and the needle to C, and the reading d say taken. The quadrants remaining as they were, the needle is connected to D, and the reading d' taken. Now if at any instant the potential of A be V_1, of B V_2, of C_1 V_1', and of D_1 V_2', we get by (17) above

$$d = \frac{k}{\tau}\int_0^\tau (V_1 - V_2)\left(V_1' - \frac{V_1 + V_2}{2}\right) dt$$

$$d' = \frac{k}{\tau}\int_0^\tau (V_1 - V_2)\left(V_2' - \frac{V_1 + V_2}{2}\right) dt$$

and by subtraction and division by $k R$

$$\frac{d - d'}{k R} = \frac{1}{\tau R}\int_0^\tau (V_1 - V_2)(V_1' - V_2')\,dt \quad . \quad (35)$$

But we have seen that the expression on the right hand side of (35) is the true mean value of the activity required.

[1] A. Potier, *Journal de Physique*, t. ix. p. 227, 1881.

CHAPTER XI.

MEASUREMENT OF INTENSE MAGNETIC FIELDS.

WE have seen above (p. 25) that every element of a conductor carrying a current in a magnetic field is acted on by a force tending to move it in a direction at right angles to its length and to the direction of the resultant magnetic force at the element, and have stated how the magnitude of the force may be calculated in terms of the intensity of the field and the strength of the current. Hence if we know the strength of the current flowing in a conductor placed in a magnetic field, and measure the force exerted in virtue of electromagnetic action on any element of the conductor, we can calculate the intensity of the field at the element. On this principle are founded the following simple methods, due to Sir William Thomson, of determining in absolute measure the intensity of magnetic fields in dynamo machines or other electromagnetic apparatus.

We shall take first the case of two long straight pole faces oppositely magnetized and placed at a short distance apart, facing one another, with their lengths vertical. In the middle of the space between the poles a wire, w (Fig. 14), somewhat longer than the poles, so as to extend a little above and below them, is hung vertically, by a cord of four or five feet in length, attached near its upper end from a fixed

peg above, and is stretched by the weight, W, attached near its lower end. Two pendulums, made of weights, $P_1 \ P_2$, as bobs carried by fine threads, are hung from two sliding pieces which can be moved along a graduated cross-bar S_1
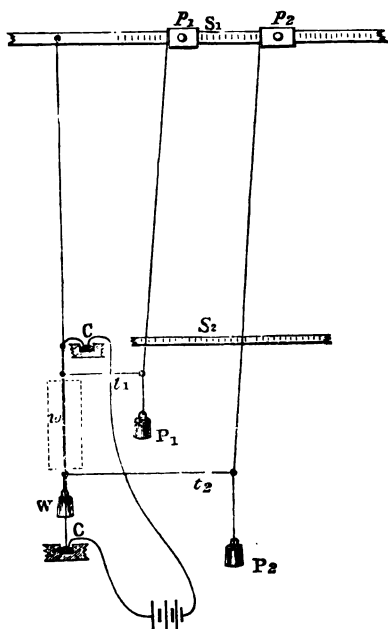
FIG. 14.

above, so placed that the ends are as nearly as possible in a plane parallel to the pole faces passing through the middle of the space between them. The two pendulum threads and the wire, w, are thus nearly in one plane. One of

these pendulums is made so long as to have its bob below the level of the lowest part of the pole faces, while the other has its bob a little below the level of the top of the pole faces, and the former is placed at the greater distance from the suspended wire. A thin thread attached to the upper end of the suspended wire is carried out horizontally and made fast at its other end to the suspension thread of the nearer pendulum.

A similar thread is attached at one end to a point of the wire near the bottom of the pole faces, and carried out similarly and made fast at the other end to a point nearly on the same level in the suspension thread of the further pendulum. The upper and lower ends of the wire, w, are placed, as shown, in mercury cups, to which are also connected the electrodes of a battery, by means of which a current can be sent through the wire w, and measured by means of a galvanometer in the circuit. A scale, S_2, is placed a little behind the plane of the threads, so that the position of a point in each, on the same level near their lower ends, can be easily read off.

When an experiment is made, the sliding pieces, $p_1 p_2$, are moved towards the left until the threads, $t_1 t_2$, are quite slack, and the positions of each thread on the upper and lower scales are read off and noted. The position of the wire, w, when $t_1 t_2$ are quite slack is also marked at the upper and lower ends of the pole faces or elsewhere. A current is then sent through the wire, w, in such a direction that the electromagnetic force acting on it moves it towards the left. The sliding pieces, $p_1 p_2$, are then moved towards the right so as to cause the pendulums to pull the wire by means of the threads, $t_1 t_2$, back again to its initial position. When the upper and lower ends have come back to their former positions, the electromagnetic forc~

on the wire is balanced by the pulls exerted by the
pendulums. The positions of the pendulum threads are
again read off on the upper and lower scales and noted
with the strength of the current flowing in w. From these
results we can easily calculate the average intensity of the
field at the place occupied by the wire, w. For let W be
the mass of each of the pendulum bobs in grammes, d the
distance through which the top of the pendulum thread
has been carried by p_1 to the right of the point of the
thread opposite to the lower scale, S_2, d_2 the correspond-
ing distance for the other pendulum, l the vertical distance
between the levels of the tops of the pendulum threads and
the lower scale measured in the same units as d_1, d_2, L the
length of the opposed pole faces, and C the strength of
the current in c.g.s. units (one-tenth the number of
amperes). The downward force in dynes on each of the
masses is Wg, where g is the acceleration in centimetres
per second per second (= 981·4 in latitude of Glasgow)
produced by gravity in a falling body at the place of
experiment. The total pull to the right exerted by the
threads on the wire is therefore $Wg\dfrac{d_1 + d_2}{l}$, and this is
equal to the pull towards the left on the wire pro-
duced by the electromagnetic action. If I be the
average intensity of the field along the wire in c.g.s. units,
we have for this pull in dynes ILC. Hence we get the
equation

$$I L C = W g \frac{d_1 + d_2}{l},$$

and therefore,

$$I = \frac{W g}{L C} \cdot \frac{d_1 + d_2}{l}. \quad \ldots \quad (1)$$

In an experiment made on September 16, 1882, with a

similar arrangement, W was 100 grammes, l 100 centi-
metres, C ·188 in c.g.s. units of current, L 30 centimetres,
and $d_1 + d_2$ 25·84 centimetres.   Hence,

$$I = \frac{100 \times 981 \cdot 4}{30 \times \cdot 188} \cdot \frac{25 \cdot 84}{100} = 4493.$$

The wire, w, should not be so flexible as to bend per-
ceptibly under the influence of the forces to which it is
subjected, so that the value of I found may be nearly
enough the average value of the intensity along a straight
line in the space between the pole faces.

In cases in which, as in many dynamo machines, the
opposite pole faces of the electromagnets are at a con-
siderable distance apart, with or without pieces of soft
iron in the intermediate space, it is practically useful to
find simultaneously the magnetic field intensity along two
lines in the same plane, one in the vicinity of each pole
face.   This may be done by so placing the electromagnets
that the two lines along which the field is measured
are in a horizontal plane, and using, instead of the
single wire carrying the current, a rectangle of copper
wire, or strip, of which the opposite sides are in these lines,
supported on knife-edges in the bisecting line parallel to
the pole face so that it can turn round that line as axis.
The frame should be weighted symmetrically on the two
sides of the line of knife-edges, so that it rests with just
enough of stability in the horizontal position.   The ends
of the wire or strip forming the rectangle are brought out
one above the other at one of the knife-edges with a
piece of insulating material between them, and bent over
so that the end of each dips into a mercury-cup in line
with the knife-edges.   The electrodes of a battery are
connected to the mercury-cups and a measured current is
sent round the rectangle.   Since the poles have opposite

magnetisms, the electromagnetic action causes one side of the rectangle to move upwards, the other side to move downwards, and thus turns the rectangle round the knife edges.

The moment of the electromagnetic forces is balanced by the action of weights, which may be riders of known weight made of wire, placed on the sides of the rectangle which is thus brought back to its initial position. If we call I the average intensity of the fields along the two sides of the rectangle in the equilibrium position, and C the current strength, both as before measured in c.g.s. units, L the length of each side, and d the distance between them in centimetres, the moment of the electromagnetic forces round the knife-edges is $I C L d$. The opposite moment resisting the motion is, if only one weight of W grammes at a distance of d' cms. from the line of knife-edges is used, $W g d'$. Hence, equating these moments, we get

$$I = \frac{W g}{C L} \frac{d'}{d}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (2)$$

from which I can be calculated. If more than one weight, W, is used, each must be multiplied by its distance from the line of knife-edges, and the sum of the products multiplied by g for the equilibrating moment.

In some cases it may be convenient to use more than one turn of wire in the rectangle. If there be n turns, each of length L, $n L$ is to be used instead of L in the formula above.

An obvious modification of this arrangement, which may be useful in some cases, is a rectangle suspended in a vertical plane, and kept in equilibrium in the proper position when no current is flowing through it, by means of a bifilar suspension, or a single thread or thin wire

under torsion. When a current is sent through the frame, it is deflected round a vertical axis by the electromagnetic action, and is brought back to the initial position of equilibrium by means of two pendulums, the points of suspension of which are on sliding pieces which can be moved along horizontal parallel bars fixed above at right angles to the plane of the rectangle when in the equilibrium position, and in the same vertical planes as its sides. Each pendulum cord has attached to it a thread which pulls horizontally at the middle of one side of the rectangle. When the rectangle is deflected, the sliding pieces are moved in opposite directions, so that, in consequence of the opposite inclinations of the pendulums to the vertical, forces restoring equilibrium are applied to the rectangle. As before, we have for the electromagnetic couple $I\,C\,L\,d$. Supposing the two points of suspension of the pendulums to be on one level, and the points of attachment of the pulling threads to the pendulum cords to be on a level lower by a distance of l cms., the distances of the verticals through the points of suspension from the corresponding verticals through the attachments of the threads to the pendulum cords to be d_1, d_2 cms. for the respective pendulums, and W grammes the mass of each bob, we have, for the moment of the equilibrating forces, the value $W g \cdot \dfrac{d_1 + d_2}{l} d$. Hence, equating moments, we get

$$I = \frac{W g}{C L} \cdot \frac{d_1 + d_2}{l} \quad \cdot \quad \cdot \quad \cdot \quad (3)$$

If $I L$ is the same for both sides of the rectangle, d_1 and d_2 will be equal; but in general there will be a small difference between the two values.

In some important practical cases the pole faces are of small area and are at only a small distance apart. If ther-

is room, a small rectangular coil, similar to that of a siphon recorder (see Fig. 15), but of comparatively few turns of wire, and without an iron core, may be hung, as described above, between the poles, with its plane parallel to the lines of force, by a bifilar or a torsion thread or wire, and a measured current sent through it. A rigid projecting arm fixed to the coil at the middle of its upper end and at right angles to the plane of the coil, has resting against it the suspension thread of a pendulum, attached at its upper end to a sliding piece movable along a horizontal bar carrying a millimetre scale, above and at right angles to the projecting arm ; and by this means the coil is brought back to the initial position. When no current is flowing through the coil, the thread is allowed to hang vertically just touching the bar and the reading on the scale above noted. Let the difference between this reading and that obtained when the pendulum is deflected be d, and let l be the vertical height of the point of suspension above the projecting arm. The horizontal force exerted by the pendulum is $Wg\dfrac{d}{l}$, and the moment of this round the vertical axis about which the coil turns is $Wgr\dfrac{d}{l}$, where r is the distance of the pendulum thread from the central plane of the coil. If n be the number of turns in the coil, b cms. its mean breadth, and L cms. the mean length of each side, the moment of the electromagnetic forces is $n\,b\,I\,L\,C$. We have, therefore,

$$I = \frac{W\,g\,r\,d}{n\,L\,C\,b\,l} \quad . \quad . \quad . \quad . \quad (4)$$

This method has frequently been used for the determination of the magnetic field intensity of the magnets of siphon recorders. The coil hanging in its place was

used as the measuring coil, and when no current was flowing through it, was kept hanging vertically in stable
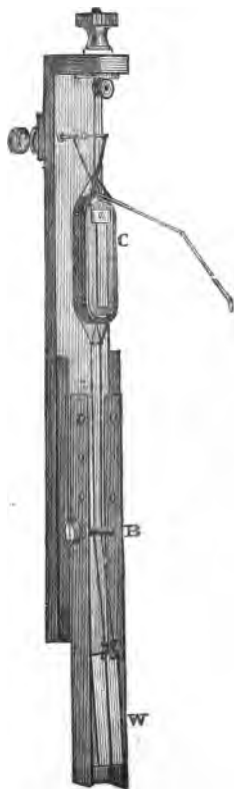
FIG. 15.

equilibrium with its plane parallel to the lines of force by

the bifilar threads attached beneath it. These threads were kept taut and bearing against the bridge B by the weights W, resting on a plane slightly inclined to the vertical. A current from one or two cells was then sent through the coil, and the difference of potentials between the terminals of the coil measured by means of a potential galvanometer. The thread of the pendulum was made to pull against the projecting aluminium arm to which the siphon is attached as shown in the figure, so as to bring the coil back to the initial position. The value of d was then read off, and that of C deduced from the known resistance of the coil and the result of the measurement with the galvanometer, and being substituted with those of the other quantities $W, n, b,$ &c. in (4), gave the value of I.

The following method, which has been frequently used in the Physical Laboratory of the University of Glasgow, is very convenient and useful in many cases. It consists in exploring the magnetic field by means of the induced current in a wire moved quickly across the lines of force over a definite area in the field. The wire is in circuit with a reflecting "ballistic" galvanometer—that is, a galvanometer the system of needles of which has so great a moment of inertia that the whole induced current due to the motion of the wire has passed through the coil before the needle has been sensibly deflected. The deflection thus obtained is noted, and compared with the deflection obtained when, with the same or a smaller resistance in circuit, a portion of the conductor is made to sweep across the lines of force over a definite area of a uniform field of known intensity, such as that of the earth or its horizontal or vertical component.

In performing the experiments, it is necessary to take

precautions to prevent any action except that between the definite area of the field selected and the wire cutting its lines of force. For this purpose the conducting-wire, which is covered with insulating material, is bent so as to form three sides of a rectangle, the middle one of which is of the length of the portion of field to be swept over. This side is placed along one side of the space over which it is about to be moved so that the connecting wires lie along the ends of the space ; and the open rectangle is then moved in the direction of its two sides until the opposite side of the space is reached. The connecting wires thus do not cut the lines of force, and the induced current is wholly due to the closed end of the rectangle.

Instead of a single wire cutting the lines of force, a coil of proper dimensions (for many purposes conveniently of rectangular shape), the mean area of which is exactly known, may be suspended in the field with its plane parallel to the lines of force, and turned quickly round through a measured angle of convenient amount not exceeding 90° ; or it may be suspended with its plane at right angles to the lines of force and turned through an angle of 180°. If n be the number of turns, A their mean area, and I the mean intensity of the field over the area swept over in each case, then, in the first case, if θ be the angle turned through, the area swept over is $n A \sin \theta$ and the number of lines cut is $n I A \sin \theta$; in the second, the area is $2 n A$, and the number of lines cut is $2 n I A$.

In order that with the feeble intensity of the earth's field a sufficiently great deflection for comparison may be obtained, it is necessary that a relatively large area of the field should be swept over by the conductor. One convenient way is to mount on trunnions a coil of moderately

fine wire of a considerable number of turns wound round a ring of large radius, like the coil of a standard tangent galvanometer, and arranged with stops so that it can be turned quickly round a horizontal axis through an exact half-turn, from a position in which its plane is exactly at right angles to the dip. This coil, if the ballistic galvanometer is sensitive enough, may always remain in the circuit. The change in the number of lines of force passing through the coil in the same direction relatively to the coil, produced by the half-turn, is plainly equal to twice as many times the area of the turn of mean area as there are turns in the coil (the effective area swept over) multiplied by the total intensity of the earth's magnetic force at the place of experiment. Or, and preferably when the horizontal component of the earth's magnetic force has been determined by experiment, the coil may be placed in an east and west (magnetic) vertical plane, and turned through an exact half-turn. The magnetic field intensity by which the effective area is to be multiplied is in this case the value of H.[1]

A sufficiently large area of the earth's field for comparison may, in some cases, be obtained very readily by carrying the wire along a rod of wood, say two or three metres long, and suspending this rod in a horizontal position by the continuations of the conductor at its ends from two fixed supports in a horizontal line at a distance apart equal to the length of the rod, and securing the remaining wires in circuit so that they may not cause disturbance by their accidental motion. The rod will thus be free to swing like a pendulum by the two suspending wires. The

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[1] The method of reducing results of observations to absolute measure by means of an earth inductor was used by Professor W. H. Rowland in his experiments on the magnetic permeability of iron, steel, and nickel.—*Phil. Mag.*, vol. 46, 1873.

pendulum thus made is slowly deflected from the vertical until it rests against stops arranged to limit its motion. When the needle is at zero, the rod is quickly thrown to the other side against similar stops there, and caught· The straight .conductor thus sweeps over an area of the vertical component of the earth's field equal to the product of the length of the rod into the horizontal distance between the two positions of the conductor at the extremities of its swing. The rod may be placed at any azimuth, as the suspending portions of the conductor in circuit, moving in vertical planes, can cut only the horizontal lines of force ; and the induced currents thus produced have opposite directions and neutralize one another.

The calculation of the results is very simple. By the theory of the ballistic galvanometer (the same *mutatis mutandis* as that of the ballistic pendulum), if q be the whole quantity of electricity which passes through the circuit, and if θ be the angle through which the needle has been deflected, or the " throw," we have, neglecting air resistance, &c.,

$$q = \frac{2}{G} \sqrt{\frac{\mu H}{m}} \sin \frac{\theta}{2};$$

where μ is the moment of inertia of the needle and attachments, m the magnetic moment of the needle, H the earth's horizontal magnetic force, and G the constant of the galvanometer. If θ be small, as it generally has been in these experiments, we have

$$q = \frac{1}{G} \sqrt{\frac{\mu H}{m}} . \theta,$$

and the quantities of electricity produced by sweeping over two areas, A and A', are directly as the deflections.

M

Let A be the total area swept over in the field or portion of field the mean intensity I of which is being measured, A' and I' the same quantities for the known field, R, R' the respective total resistances in circuit, q, q' the quantities of electricity generated in the two cases, θ, θ' the corresponding deflections supposed both small ; we have

$$q = \frac{A I}{R} = \frac{1}{G} \sqrt{\frac{\mu H}{m}} \, \theta,$$

$$q' = \frac{A' I'}{R'} = \frac{1}{G} \sqrt{\frac{\mu H}{m}} \, \theta'$$

and therefore

$$I = \frac{A' R \theta}{A R' \theta'} I' \quad . \quad . \quad . \quad . \quad (9)$$

If convenient, θ and θ' may be taken as proportional to the number of divisions of the scale traversed by the spot of light in the two cases.

The error caused by neglecting the effect of air resistance, &c., in diminishing the deflection will be nearly eliminated if R and R' be chosen so that θ and θ' are nearly equal.

The following new method of reducing ballistic observations to absolute measure has been given by Sir William Thomson. A short induction coil wound round the centre of an ordinary magnetizing helix, whose length is great compared with its diameter, is kept in circuit with the galvanometer. A measured current is sent through the wire of the helix, and when the needle is at rest the circuit of the helix is broken, and the galvanometer deflection read off. If N be the number of turns of wire per cm. on the helix, C the current in electromagnetic c.g.s. units, the magnetic force within it is $4 \pi n C$ parallel to the axis ; and if A' be the proper mean area of the cross-section of the helix, and

n' the number of turns in the induction coil, the number of lines (unit tubes) of force passing out of the galvanometer circuit when the current is stopped is $4 \pi N n' A' C$ (see Note B). Hence, R' denoting the total resistance in circuit, the total quantity q' of electricity generated is $\dfrac{4 \pi N n' A' C}{R'}$, and instead of (9) we get

$$I = 4 \pi N n' C \frac{A' R \theta}{A R' \theta'} . \quad . \quad . \quad (10)$$

The ballistic method of investigation was also used by Prof. W. H. Rowland (*Phil. Mag.* vol. l., 1875) for the determination of the distribution of magnetism in magnets. A thin ring of wire was made just large enough to pass round the magnet experimented on, and was placed in circuit with a ballistic galvanometer. It was then, while encircling the magnet and held with its plane at right angles to the axis of the magnet, slided quickly along the magnet through equal short distances, and the deflection of the needle noted for each motion. The deflections thus obtained gave for thin magnets an approximate comparative estimate of the density at different points along the magnet of the surface distribution of ideal magnetic matter by which the action of the magnet could be produced, and the results were reduced to absolute measure by means of an earth-inductor.

This method, used along with Sir William Thomson's method of reduction to absolute measure, gives a very ready means of estimating with much exactness the total quantity of imaginary magnetic matter in one pole or one end of a magnet, whether of bar, horse-shoe, or other shape. The ring, which for the present purpose may be larger, and thick enough to contain any convenient number of turns, is placed at the centre or nearly neutral region of

the magnet, and then quickly pulled off and away from the magnet, and the galvanometer deflection (θ) noted. A measured current is then sent through the helix, and the deflection (θ') produced by suddenly opening the circuit of the helix also observed. Let n be the number of turns in the ring of wire, and ϕ the total quantity in c.g.s. units of imaginary magnetic matter in the portion of the magnet swept over, then the number of lines of force cut through by each turn of wire in the ring is $4 \pi \phi$, (see Note B), and if R be the total resistance in circuit, the total quantity (q) of electricity generated is $\dfrac{4 \pi n \phi}{R}$. We have therefore

$$q = \frac{4 \pi n \phi}{R} = G \sqrt{\frac{\mu H}{m}} \sin\frac{\theta}{2},$$

and for the helix we get from the calculation above

$$q' = \frac{4 \pi N n' A' C}{R'} = \frac{1}{G} \sqrt{\frac{\mu H}{m}} \sin\frac{\theta'}{2}.$$

By division we get

$$\phi = N A' C \frac{n' R \sin\dfrac{\theta}{2}}{n R' \sin\dfrac{\theta'}{2}} \quad . \quad . \quad . \quad (11)$$

and if the deflections are small angles,

$$\phi = N A' C \frac{n' R \theta}{n R' \theta'}. \quad . \quad . \quad (12)$$

This equation is of course also applicable to the reduction to absolute measure of the results of determinations of magnetic distribution made by the ballistic method. The value of ϕ deduced for each deflection divided by the area of the corresponding small portion of the magnet is approximately the surface density of the ideal distribution, the distribution on the end faces being of course included in the end deflections.

CHAPTER XII.

THEORY OF THE DIMENSIONS OF THE UNITS OF PHYSICAL QUANTITIES.

WE have, in p. 43 above, explained the term *change-ratio* of a physical quantity : in order to show clearly the relations of the various absolute units of electrical and magnetic measurement to the units on which they are based, we shall here investigate for each of the principal quantities the formula of dimensions from which the numerical value of the change-ratio is to be found in any particular case.

A physical quantity is expressed numerically in terms of some convenient magnitude of the same kind taken as unit, and compared with it. This unit is of course capable itself of being expressed numerically in terms of any unit of the same kind. Hence if Q be the numeric [1] of any physical quantity in terms of one unit, Q' in terms of another unit, and n, n' the numerics of the respective units in terms of any unit whatever of the same kind, we

[1] The term numeric has been introduced by Prof. James Thomson (Thomson's "Arithmetic" Ed., LXXII., p. 4) as an abbreviation of "numerical expression." It denotes a number, or a proper fraction, or an improper fraction, or an incommensurable ratio. We shall find it convenient to employ it here where we wish to lay stress on the fact that we are dealing with what are essentially numerical expressions. Of course what is actually meant by the conveniently brief expressions "a length L," "a mass M," "a force F," and the like, is simply that L, M, F, &c., denote the corresponding numerics.

have, since the numerics of the quantity in terms of the
third unit must be equal, the equation

$$n\,Q = n'\,Q',$$

and therefore,

$$Q' = \frac{n}{n'}\,Q \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In order therefore to find the expression Q' of the magni-
tude in terms of the second unit from its expression in
terms of the first unit, we have to multiply by the
ratio $\frac{n}{n'}$, that is, in fact, by the numeric of the first unit of
measurement in terms of the second. The numeric which
expresses the ratio $\frac{n}{n'}$ for a change from one given unit of
measurement of the quantity to another given unit is
therefore appropriately called the *change-ratio* of the
quantity for that change.

   In order therefore that we may be able to find the
change-ratio in any given case of change of units, we
must know the general form which $\frac{n}{n'}$ has for the quantity
considered, that is, we must know how it involves the
corresponding ratio for each of the fundamental units.
The formula which expresses $\frac{n}{n'}$ for a unit of measure-
ment of any quantity we shall call the *formula of
dimensions* or the *dimensional formula* of the unit of
that quantity, or simply the dimensional formula of the
quantity. To prevent the necessity for the constant
repetition of these terms we shall denote the dimensional
formula of the unit of a quantity of which the numeric
is denoted by any particular symbol, by the same symbol

inclosed in square brackets. Thus we denote the dimensional formula of the quantity Q by the symbol $[Q]$.

For example, if we wish to find from the numeric of a length in terms of the yard as unit, the numeric of the same length in terms of the metre as unit, the ratio $[Q]$ becomes $\dfrac{36}{39\cdot37043}$, the ratio of the expressions of the two units in terms of the inch as units of comparison ; or $\dfrac{91\cdot439}{100}$, the same ratio in terms of the centimetre as unit ; or simply $\cdot91439$, the same ratio in terms of the metre as unit.

All complex physical quantities are expressed in accordance with their definitions in terms of units which involve in a more or less complex manner the fundamental units of Length, Mass, and Time. These are called *derived units*.

FUNDAMENTAL UNITS.—(1) *Length.* We have seen above how to find the change-ratio for numerics of lengths. It is usual to denote the dimensional formula in this case by the symbol $[L]$.

(2) *Mass.* The dimensional formula for the unit of mass is denoted by $[M]$.

(3) *Time.* Similarly the dimensional formula for the unit of time is denoted by $[T]$.

DERIVED UNITS.—Let us suppose that the numeric Q of a physical quantity is given by the equation,

$$Q = C\, L_1^{\,l}\, M_1^{\,m}\, T_1^{\,n}.\ \ L_2^{\,s}\, M_2^{\,q}\, T_2^{\,r}.\ \&c., \qquad . \quad (2)$$

where L_1, L_2, &c., M_1, M_2, &c., T_1, T_2, &c., are numerics of different lengths, masses, and times in terms of a certain chosen unit for each, and C is a numerical multiplier which does not depend on the units chosen. Now let other units of length, mass, and time be chosen,

and let Q' be the numeric of the same quantity in terms of these units, and L'_1, L'_2, &c., M'_1, M'_2, &c., T'_1, T'_2, &c., those of the lengths, masses, and times. Then we have

$$Q' = L'^l_1 M'^m_1 T'^n_1 . L'^p_2 M'^q_2 T'^r_2 . \&c.$$

But by equation (1) $L'^l = L^l [L]^l$, $M'^m = M^m [M]^m$, and so on. Hence (3) becomes,

$$Q' = L_1^l M_1^m T_1^n . L_2^p M_2^q T_2^r . \&c. [L]^{l+p+\&c}.$$
$$[M]^{m+q+\&c}. [T]^{n+r+\&c} \quad . \quad . \quad . \quad (4)$$

By equation (1) therefore the dimensional formula $[Q]$ of the quantity is $[L]^{l+p+\&c}. \ [M]^{m+q+\&c}. \ [T]^{n+r+\&c}.$ In accordance with the notation $[Q]$, we shall denote this in future by the more convenient expression $[L^{l+p+\&c}. \ M^{m+q+\&c}. \ T^{n+r+\&c}]$.

The numerics $l + p + \&c.$, &c., correspond to what Fourier (*Théorie Analytique de la Chaleur*, Chap. II., Sect. IX.) called *les exposants des dimensions* of the quantities which entered into his analysis, and it is these numerics, not the dimensional formulas, which are properly the "dimensions" of the units. It was pointed out by Fourier that in equations involving the numerics of physical quantities every term must be of the same dimensions in each unit, otherwise some error must have been made in the analysis. This consideration affords in physical mathematics a valuable check on the accuracy of algebraical work.

It is obvious from equations (1) or (4) that the dimensional formula of the product of any number of numerics of different physical quantities Q_1, Q_2, is the product $[Q_1 . Q_2. \&c.]$ of their dimensional formulas, and more generally that the dimensional formula of the product $Q_1^{\mu_1}$, $Q_2^{\mu_2}$, &c., of any powers whatever of these expressions, is the product of the same powers of the corresponding dimensional formulas.

We are now prepared to find the dimensional formulas of the various derived units. The process will consist in finding for each quantity the formula corresponding to the right-hand side of (2), and in thence deriving according to (4) the proper formula of dimensions. We shall consider first the units of Area, Volume, and Density; then the various dynamical units which are involved in those of electrical and magnetic quantities.

Area. The general formula for the area of any surface can be put in the form $C L^2$, where L is a numeric expressing a length, and C is a numeric which does not change with the units. Hence by (4) the formula of dimensions for area is $[L^2]$.

Volume. Similarly the formula for the numeric of a volume can be written $C L^3$, and the formula of dimensions is $[L^3]$.

Density. The density of a homogeneous body is measured by the numeric of the mass per unit of volume. Hence, if the numeric of the mass in a volume $C L^3$ be M, the numeric of the density is given by the formula $\dfrac{M}{C L^3}$, and the dimensional formula is $[M L^{-3}]$.

DERIVED DYNAMICAL UNITS.—*Velocity* [s]. The velocity of a body is measured by the numeric of the length described per unit of time. If the body move uniformly and describe L units of length in T seconds, its velocity is evidently given by the formula $\dfrac{L}{T}$, and therefore [s] is $[L T^{-1}]$.

Acceleration [s]. The acceleration of a body is measured by the numeric of the change of velocity per unit of time. If the acceleration be uniform and $\dfrac{L}{T}$ express the increase or diminution of velocity, and T' the time ⁚

which it is effected, the formula evidently is $\frac{L}{T} \cdot \frac{1}{T}$, and $[\ddot{s}]$ is $[L\,T^{-2}]$.

Momentum. The momentum of a body is measured by the product of the numerics of its mass and velocity. The formula is therefore $M\frac{L}{T}$, and the dimensional formula is $[M\,\dot{s}]$ or $[M\,L\,T^{-1}]$.

Force $[F]$. A force is measured by the numeric of the momentum which it generates per unit of time. The formula is accordingly $\frac{M L}{T} \cdot \frac{1}{T}$, and therefore $[F]$ is $[M\,L\,T^{-2}]$.

Work $[W]$. The work (p. 45) done by a force is measured by the product of the numeric F of the force, and that of the space L through which it has acted. The formula for this numeric is therefore $F L$. The dimensional formula $[W]$ is therefore (p. 166) $[F L]$ or $[M\,L^2\,T^{-2}]$.

Activity $[A]$. Activity (p. 46) is measured by the numeric of the work done per unit of time. Its dimensional formula $[A]$ is therefore $[W\,T^{-1}]$ or $[M\,L^2\,T^{-3}]$.

We shall here, for the sake of illustration, give three examples of the application of dimensional formulas to the solution of problems regarding units. The problems are taken from Professor Everett's *Units and Physical Constants*.

Ex. 1. If the unit of time be the second, the unit density 162 lbs. per cubic foot, and the unit of force the weight of an ounce at a place where the acceleration g produced by gravity in one second is 32 feet per second, what is the unit of length?

Here the change-ratio by which we must multiply the numeric of the density of a body in the system of units

proposed, to find its density in terms of the pound as unit of mass, and the foot as unit of length, is 162. We have therefore, omitting the brackets in the dimensional formulas

$$M L^{-3} = 162$$

where M is the number of pounds equivalent to the unit of mass, and to the number of feet equivalent to the unit of length. Also, it is plain that the unit of force in the proposed system is two foot-pound-second units. Hence we have also since $T = 1$.

$$M L T^{-2} = M L = 2.$$

By division therefore we get $L^4 = \dfrac{1}{81}$ or $L = \dfrac{1}{3}$. The unit of length is therefore 4 inches.

Ex. 2. The number of seconds in the unit of time is equal to the number of feet in the unit of length, the unit of force is 750 lbs. weight (g being 32), and a cubic foot of the substance of unit density contains 13500 ounces. Find the unit of time.

Using M and L as in the last problem, and putting T for the number of seconds equivalent to the unit of time we have plainly

$$M L^{-3} = \frac{13500}{16}$$

and

$$M L T^{-2} = 750 \times 32.$$

Therefore by dividing and remembering that $L = T$, we get

$$T^2 = \frac{32 \times 750 \times 16}{13500} = \frac{16^2}{3^2}$$

or

$$T = \frac{16}{3},$$

That is the unit of time is $5\frac{1}{3}$ seconds.

Ex. 3. When an inch is the unit of length and T seconds the unit of time the numeric of a certain acceleration is a; when 5 feet and 1 minute are the units of length and time respectively, the numeric of the same acceleration is $10a$. Find T.

The change-ratio or value of $L\,T^{-2}$ for reduction to foot-second units is plainly in the first case $\dfrac{1}{12}\,T^{-2}$, in the second $\dfrac{5}{3600}$. We get therefore

$$\frac{1}{12}\,T^{-2}\,a = \frac{5}{3600} \times 10\,a$$

or

$$T = \sqrt{6}.$$

DERIVED ELECTRICAL AND MAGNETIC UNITS— ELECTROSTATIC SYSTEM.—*Quantity of Electricity* $[q]$. In the electrostatic system the units of all the other quantities are founded on the following definition of unit quantity of electricity, which is precisely similar to the definition of magnetic pole (p. 20) which forms the basis of the electromagnetic system. *Unit quantity of electricity is that quantity which concentrated at a point at unit distance from an equal and similar quantity also concentrated at a point is repelled with unit force.*[1] Hence by Coulomb's laws that electric attractions and repulsions are directly as the product of (the numerics of) the attracting or repelling quantities, and inversely as the second power of (the numeric of) the distance between them, if a quantity of positive electricity expressed by e be placed at a point distant L units from an equal quantity of positive electricity, the numeric F of the force between them is $\dfrac{q^2}{L^2}$. We have therefore the

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[1] The medium between the points is supposed to be air.

equation $q^2 = F L^2$, and therefore $[q]$ is $[F^{\frac{1}{2}} L]$ or $[M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1}]$.

Electric Surface Density $[\sigma]$. The density of an electric charge is measured by the quantity of electricity per unit of area. Therefore $[\sigma]$ is $[q L^{-2}]$ or $[M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1}]$.

Electric Force and Intensity of Electric Field $[i]$. The electric force at any point in an electric field is the force with which a unit of positive electricity would be acted on if placed at the point. Hence if the numeric of the quantity of electricity at a point P be q, and that of the electric force at that point be i, the numeric F of the force on the electricity is $q\,i$, and we have the equation $i = F q^{-1}$. Therefore $[i]$ is $[F q^{-1}]$ or $[M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1}]$.

The intensity of an electric field at any point is measured by the electric force at that point, and therefore has the same dimensional formula.

Electric Potential $[v]$. The difference of potential between two points is measured by the work which would be done if a unit of positive electricity were placed at the point of higher potential and made to pass by electric forces to the point of lower potential. Hence in transferring q units of electricity through a difference of potentials expressed by v, an amount of work is done of which the numeric W is $q\,v$. We have therefore $v = W q^{-1}$, and hence $[v]$ is $[W q^{-1}]$ or $[M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1}]$.

Capacity of a Conductor $[c_p]$. The capacity of an insulated conductor is the quantity of electricity required to charge the conductor to unit potential, all other conductors in the field being supposed at zero potential. Hence, denoting the numeric of the capacity of a given conductor by c_p, those of its charge and potential by Q and V respectively, we have $c_p = Q V^{-1}$, and for $[c_p]$ therefore $[Q V^{-1}]$, that is $[L]$. The unit of capacity has

therefore the same dimensions as the unit of length; and the capacity of a conductor is properly expressed in c.g.s. electrostatic units as so many centimetres.[1]

Specific Inductive Capacity $[k]$. The specific inductive capacity of a dielectric is the ratio of the capacity of a condenser, the space between the plates of which is filled with the dielectric, to the capacity of a precisely similar condenser with air as dielectric; or, according to Maxwell's Theory of Electric Displacement,[2] it is defined as the ratio of the electric displacement produced in the dielectric to the electric displacement produced in air by the same electromotive force. It is therefore in the electrostatic system simply a numerical coefficient which does not change with the units. Hence $[k] = 1$.

Electric Current $[c]$. An electric current in a conducting wire is measured by the quantity which passes across a given cross-section per unit of time. If q be the numeric of the quantity which has passed in a time of which the numeric is T, then denoting the numeric of the current by c, we have $c = \dfrac{q}{T}$, and $[c]$ is $[q \, T^{-1}]$ or $[M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2}]$.

Resistance $[r]$. By Ohm's law the resistance of a conductor is expressed by the ratio of the numeric r of the difference of potentials between its extremities to the numeric c of the current flowing through it. We have therefore $r = \dfrac{v}{c}$, and $[r]$ is $[v \, c^{-1}]$ or $[L^{-1} T]$.

Conductivity. The change-ratio of conductivity is plainly $[L \, T^{-1}]$. The change-ratio for conductivity in

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[1] The electrostatic capacity of a spherical conductor is numerically equal to the radius of the conductor. The absolute c.g.s. unit of electrostatic capacity in electrostatic measure is the capacity of a spherical conductor 1 cm. in radius at an infinite distance from all other bodies.

[2] Maxwell's *Electricity and Magnetism*, vol. i. p. 133.

electrostatic measure is thus the same as that for velocity. Hence a conductivity in electrostatic c.g.s units is properly expressed in centimetres per second.

The following illustration of this result has been given by Sir William Thomson. Suppose a spherical conductor charged to a potential v to be connected to the earth by a long thin wire, of which the capacity may be neglected; and let r be the resistance of this wire in electrostatic measure. The current in the wire at the instant of contact is $\frac{v}{r}$. Now let the sphere diminish in radius at such a constant rate that the potential remains v. The current remains $\frac{v}{r}$, and the quantity of electricity which flows out in t seconds will be $\frac{v}{r} t$. If the radius be initially x, and in t seconds has diminished to x', the diminution of capacity is (footnote, p. 172) $x - x'$. Hence the loss of charge is $v (x - x')$, and we get $\frac{v}{r} t = v (x - x')$, or $\frac{1}{r} = \frac{x - x'}{t}$. But $\frac{x - x'}{t}$ is the velocity with which the radius of the sphere diminishes. The conductivity $\frac{1}{r}$ of the wire is therefore measured numerically by the velocity with which the surface of the sphere must approach the centre, in order that its potential may remain constant when the surface is connected to the earth through the wire.

ELECTROMAGNETIC SYSTEM.—*Magnetic Pole* or *Quantity of Magnetism* [m]; *Surface Density of Magnetism* [σ']; *Magnetic Force or Magnetic Field Intensity* [I]; *Magnetic Potential* [v'].

The electromagnetic system of units is based on the unit magnetic pole as defined above (p. 20). This definition is

exactly the same as that of unit quantity of electricity on which the electrostatic system is founded ; and therefore the purely magnetic quantities here mentioned, which bear the same relations to the unit quantity of magnetism that the corresponding electric quantities bear to the chosen unit quantity of electricity, have in the electromagnetic system the same dimensional formulas as those just found for the latter quantities in the electrostatic system.

Magnetic Moment [*m*]. The numeric of the magnetic moment of a uniformly magnetized bar-magnet is the product of the numerics of the strength of either pole and the length of the magnet. Hence we have $[m] = [M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2}] \cdot [L] = [M^{\frac{1}{2}} L^{\frac{5}{2}} T^{-2}]$.

Intensity of Magnetization [*ν*]. The intensity of magnetization of any portion of a magnet is measured by the magnetic moment of that portion per unit of volume. Hence, if *ν* denote the numeric of the intensity of magnetization of a uniformly magnetized magnet, the numerics of the magnetic moment and volume of which are *m* and $A L^3$, we have $\nu = \dfrac{m}{A L^3}$, and $[\nu] = [M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-2}]$.

It is plain that the intensity of magnetization of a uniformly and longitudinally magnetized bar is equal to the surface density of the magnetic distribution over the ends of the bar, and therefore intensity of magnetization has the same dimensional formula as magnetic surface density.

Magnetic Permeability [1] [*κ*]. The magnetic permeability of an inductively magnetized substance, or its magnetic inductive capacity, is the analogue in magnetism of specific inductive capacity of a dielectric in electricity, and of the conductivity of a body for heat in heat conduction. It is measured at any point by the

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[1] See Reprint of Papers on *Electrostatics and Magnetism*, by Sir W. Thomson, p. 484.

ratio of the numeric of the force which a unit pole would there experience if placed in a narrow crevasse, cut in the substance so that its walls are at right angles to the direction of magnetization, to that of the force which it would experience if placed in a narrow crevasse, the walls of which are parallel to the direction of magnetization. In the first case we must suppose, in consequence of the formation of the crevasse, a distribution of blue magnetism over one of its walls, and a similar distribution of red magnetism over the opposite wall. These magnetic distributions are exactly equal and opposite to the distributions on the sides of the portion of the substance removed to form the crevasse. If we suppose the substance uniformly magnetized, the density of this distribution will be uniform on both walls, and if we denote the density in one by σ', that on the other will be denoted by $-\sigma'$. A unit blue magnetic pole placed in the crevasse would be repelled from the blue side and attracted towards the red with a force in each case equal to $2\pi\sigma'$; [1] and therefore would be acted on towards the red

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[1] Imagine a material particle of unit mass (such a particle being here defined as a particle which placed at unit distance from an equal particle would be attracted with unit force) situated at a point on the axis of a thin uniform material disk of radius r. Let the mass of the disk per unit of area be σ, and the distance of the particle from the disk measured along the axis a. The attraction on the particle of a small element of the disk of area dS at a distance x from the centre is $\dfrac{\sigma\,dS}{a^2+x^2}$, and the component resolved along the axis is $\dfrac{a\,\sigma\,dS}{(a^2+x^2)^{\frac{3}{2}}}$. Hence the attraction in this direction due to a narrow ring of radius x and breadth dx is $\dfrac{2\pi\sigma a\,x\,dx}{(a^2+x^2)^{\frac{3}{2}}}$, and that of the whole disk therefore $2\pi\sigma a\displaystyle\int_0^r \dfrac{x\,dx}{(a^2+x^2)^{\frac{3}{2}}} = 2\pi\sigma\left(1-\dfrac{a}{\sqrt{a^2+r^2}}\right)$. If the distance a of the particle from the disk be very small in comparison with the radius r of the disk, this expression for the total attraction becomes simply $2\pi\sigma$. We have only to substitute a unit magnetic pole for the unit particle and a distribution of imaginary magnetic matter of surface density σ' for the material disk, and we have the result used in the text.

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side by a total force of $4\pi\sigma'$. Besides this force due to the magnetic distribution in the crevasse, the pole is acted on by the resultant of the forces due to the other magnetic distributions of the system. If we suppose the substance isotropic as to magnetic quality, that is, to have the same magnetic quality in different directions, this latter force will be in the same direction as the former. Calling it f, we have for the total force acting on the pole towards the red side of the crevasse the expression $f + 4\pi\sigma'$.

If, on the other hand, the pole were placed in the same position but in a narrow crevasse cut with its walls parallel to the direction of magnetization of the substance, the force acting on it would be simply f. Hence, if we call κ the magnetic permeability of the substance, we have

$$\kappa = 1 + 4\pi\frac{\sigma'}{f}.$$ But σ' and f have, as we have seen, the

same dimensional formula, and therefore the dimensional formula $[\kappa]$ is 1 in the electromagnetic system. In other words, the numeric measuring permeability is essentially independent of the fundamental units, or permeability is essentially a mere numeric.

Magnetic Susceptibility. The ratio $\dfrac{\sigma}{f}$ is called the

magnetic susceptibility of the substance. Its dimensional formula is therefore also 1 in the electromagnetic system, that is magnetic susceptibility is a mere numeric.

Current Strength $[C]$. By the theory of electromagnetic action stated above in p. 25, and the definition of unit current (3), p. 26, we have, for any actual case of a magnetic pole placed at the centre of a circle of wire

carrying a current, the equation $C = \dfrac{FL}{2\pi q}$, where F, L,

q', and C are the numerics respectively of the force acting on the pole, the radius of the circle, the strength of the pole, and the strength of the current. Hence $[C] = [F L q'^{-1}] = [M^{\frac{1}{2}} L^{\frac{1}{2}} T^{-1}]$.

Quantity of Electricity $[Q]$. The numeric Q of the quantity of electricity conveyed by a current the numeric of the strength of which is C in T seconds is equal to $C T$. Hence $[Q] = [C T] = [M^{\frac{1}{2}} L^{\frac{1}{2}}]$.[1]

Electric Potential, or Electromotive Force $[V]$. As above (p. 171), but using in this case the symbol V for the numeric of a difference of potentials, we get $W = V Q$. Thus we have $[V] = [M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2}]$.

Electrostatic Capacity. $[C_p]$. Using for the numeric of capacity in electromagnetic units the symbol C_p, we find, by the same process as in p. 172, the equation $C_p = \dfrac{Q}{V}$, $[C_p] = [L^{-1} T^2]$.

Resistance $[R]$. Using here R as the numeric of a resistance, we get as formerly $R = \dfrac{V}{C}$, and therefore $[R] = [L T^{-1}]$.

The dimensional formula for resistance is thus the same as that for velocity, and therefore a resistance in electromagnetic units is properly expressed as a velocity, and accordingly, in c.g.s. units, as so many centimetres per second. This fact is directly shown by the following illustration, due to Sir William Thomson Let the rails of the ideal machine, described in p. 40,

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1 We might pass in the electrostatic system from the dimensional formula of unit current to that of unit quantity of magnetism, precisely as we pass here in the electromagnetic system from the dimensional formula of unit quantity of magnetism to that of unit current, and we should find for the dimensional formula sought that here obtained $[M^{\frac{1}{2}} L^{\frac{3}{2}}]$. From this the formulas in the electrostatic system for all the other magnetic quantities might be found.

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be supposed to run horizontally at right angles to the magnetic meridian, and let their plane be vertical. Let a tangent galvanometer be included in the wire connecting the rails. The slider when moved along the rails will cut the lines of the earth's horizontal force, the intensity of which in electromagnetic measure we have denoted by H. If the slider have a length L and be moved with a velocity v, the electromotive force developed will be HLv. If R be the total resistance in circuit, C the current flowing, r the mean radius of the galvanometer coil, and L' the length of wire in the coil we have by (2), p. 28, $C = \dfrac{Hr^2}{L'} \tan \theta$. But by Ohm's

law $C = \dfrac{HLv}{R}$. Hence $\dfrac{HLv}{R} = \dfrac{Hr^2}{L'} \tan \theta$, or

$$R = \frac{LL'v}{r^2 \tan \theta}.$$

Now we may suppose the radius r of the coil so taken that $r^2 = LL'$, and that the slider is moved at such a speed, v, that the deflection of the needle is $45°$. Under these conditions we get $R = v$. The resistance R of the circuit is therefore measured in electromagnetic units by the velocity with which the slider must be moved, so that the deflection of the needle of the tangent galvanometer may be $45°$.

Coefficient of Self-Induction. Denoting by λ (instead of L to avoid confusion) the coefficient of self-induction of a circuit, the current in which is C, we have (p. 142) $\lambda \dfrac{dC}{dt}$ for the electromotive force of self-induction. Hence $\lambda \dfrac{dC}{dt}$ has the same dimensional formula as electromotive force, that is $[\lambda C T^{-1}] = [M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-2}]$, and therefore $[\lambda] = [L]$.

Coefficient of Mutual Induction. If μ be the co-efficient of mutual induction between two circuits, C the current in one of them, then the electromotive force in the other circuit due to mutual induction is $\mu \dfrac{d\,C}{dt}$. Hence by the same process as before we get $[\mu] = [L]$.

A coefficient of self- or of mutual induction is there-fore in electromagnetic measure simply a length, and in c.g s. units is probably expressed as so many centimetres.

We have now investigated the dimensional formulas of the absolute units of all the principal electric and magnetic quantities in the electrostatic system, or in the electro-magnetic system, according as each quantity is generally measured in practice. Each may, however, be expressed either in electrostatic or in electromagnetic units, and we give the following table of dimensional formula for all the quantities in both systems :—

FUNDAMENTAL UNITS.

Quantity.	Dimensional Formula.
Length	$[L]$
Mass	$[M]$
Time	$[T]$.

DERIVED UNITS.

I. *Dynamical Units.*

Velocity	$[L\ T^{-1}]$
Acceleration	$[L\ T^{-2}]$
Force	$[M\ L\ T^{-2}]$
Work $\Big\}$ Energy	$[M\ L^{2}\ T^{-2}]$

II. Electric Units.

	Electrostatic System.	Electromagnetic System.
Quantity of Electricity	$[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}]$	$[M^{\frac{1}{2}}L^{\frac{1}{2}}]$
Surface Density of Electricity ⎫ Electric Displacement ⎭	$[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}]$	$[M^{\frac{1}{2}}L^{-\frac{3}{2}}]$
Electric Force, or ⎫ Intensity of Electric Field ⎭	$[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}]$	$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2}]$
Electric Potential ⎫ Electromotive Force ⎭	$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}]$	$[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}]$
Electrostatic Capacity	$[L]$	$[L^{-1}T^{2}]$
Specific Inductive Capacity	I	$[L^{-2}T^{2}]$
Current Strength	$[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}]$	$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}]$
Resistance	$[L^{-1}T]$	$[LT^{-1}]$

III. Magnetic Units.

Quantity of Magnetism, or ⎫ Magnetic Pole ⎭	$[M^{\frac{1}{2}}L^{\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}]$
Surface Density of Magnetism	$[M^{\frac{1}{2}}L^{-\frac{3}{2}}]$	$[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}]$
Magnetic Moment	$[M^{\frac{1}{2}}L^{\frac{5}{2}}]$	$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2}]$
Intensity of Magnetization	$[M^{\frac{1}{2}}L^{-\frac{1}{2}}]$	$[M^{\frac{1}{2}}L^{-\frac{3}{2}}T^{-1}]$
Magnetic Force or ⎫ Intensity of Magnetic Field ⎭	$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-2}]$	$[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}]$
Magnetic Potential	$[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}]$	$[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}]$
Magnetic Permeability ⎫ Magnetic Susceptibility ⎭	$[L^{-2}T^{2}]$	I
Coefficient of Self-Induction ⎫ Coefficient of Mutual Induction ⎭	$[L^{-1}T^{2}]$	$[L]$

As an example of the use of dimensional formulas we may find the multiplier for the reduction of numerics of magnetic field intensities given in terms of British foot-grain-second units to the corresponding numerics in terms of c.g.s. electromagnetic units. Let H be the numeric of an intensity in terms of British units, H' the numeric of

the same intensity in c.g.s. units. We have, by equation
(4), p. 166,
$$I' = I[I] = I[M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1}].$$
Since 1 gramme = 15·43235 grains, and 1 centimetre
$$= \frac{1}{30\cdot47945} \text{ foot, we have}$$
$$[M^{\frac{1}{2}} L^{-\frac{1}{2}} T^{-1}] = \left(\frac{1}{15\cdot43235 \times 30\cdot47945}\right)^{\frac{1}{2}} = \frac{1}{21\cdot688}.$$
Therefore
$$H' = H \times \frac{1}{21\cdot688}.$$

The earth's horizontal force is given as 3·92 in British
units at Greenwich for 1883. We get therefore
$$H' = 3\cdot92 \frac{1}{21\cdot688} = \cdot18075$$
in c.g.s. units.

Units Adopted in Practice.

In practical work the resistances and electromotive
forces occurring to be measured are usually so great that
if the absolute electromagnetic c.g.s. units were used, the
resulting numerics would be inconveniently large ; while,
on the other hand, capacities are generally so small that
their numerics in c.g.s. units would be only very small
fractions. Accordingly certain multiples of the c.g.s. units
of resistance and electromotive force, and a submultiple of
that of capacity have been chosen for use in practice.
The first two, the ohm and the volt, together with the
practical units of current and quantity, the ampere and the
coulomb, have been explained above (Chap. V.). The
practical unit of electrostatic capacity is called the *farad*,
and may be defined as the capacity of a condenser which,
when charged by an electromotive force of one volt, has a

charge of one coulomb. If C_f be the numeric of the capacity of such a condenser in c.g.s. electromagnetic units of capacity, we have $C_f = \dfrac{10^{-1}}{10^8} = 10^{-9}$; or one farad is equivalent to 10^{-9} c.g.s.

In some cases, when the quantities to be expressed are very large, units one million times greater than the chosen practical units are employed. These are denoted by the names of the corresponding practical units with *mega* (great) prefixed. On the other hand, for the expression of very small quantities, units one million times smaller than the practical units are sometimes used, and are denoted by the corresponding names of the practical units with *micro* (small) prefixed.

Such units are however rarely employed, with the exception of the *megohm*, used for expressions of resistances of insulating substances, and the *microfarad*, which is really the most convenient unit for expressions of capacities. A megohm is a velocity of 10^{15} centimetres per second; one c.g.s. unit of capacity is equivalent to 10^{15} microfarads.

The practical units which have been adopted may be considered as belonging to an absolute system, based on a unit of length equivalent to one thousand million (10^9) centimetres (approximately the length of one quadrant of the earth's polar circumference), a unit of mass equivalent to one one-hundred-millionth of a milligramme, or 10^{-11} gramme, and the second as unit of time. The verification of this in the different cases will furnish examples of the use of dimensional formulas.

For example, let us find what the expressions of resistances and electromotive forces in c.g.s. units become when these new units of length and mass are substituted for the

centimetre and the gramme. Let R be the numeric of a resistance in c.g.s. units, and R' its numeric in terms of the new units. We have, by (2) p.177 above,

$$R' = R\,[L\,T^{-1}] = R \times \frac{1}{10^9}$$

since the unit of time remains unchanged. One ohm is therefore equivalent to 10^9 c.g.s. units of resistance, that is 10^9 centimetres per second.

Again let V be the expression of an electromotive force in c.g.s. electromagnetic units, E' its expression in terms of the new units. The dimensional formula for electromotive force is $[M^{\frac{1}{2}}\,L^{\frac{3}{2}}\,T^{-1}]$. We have therefore
$$E' = E\,[M^{\frac{1}{2}}\,L^{\frac{3}{2}}\,T^{-1}].$$

We have only to consider what $[M^{\frac{1}{2}}\,L^{\frac{3}{2}}]$ becomes. This is plainly $\left(\frac{1}{10^{-11}}\right)^{\frac{1}{2}} \times \left(\frac{1}{10^9}\right)^{\frac{3}{2}}$ or $\frac{1}{10^8}$. Hence

$$E' = E \times \frac{1}{10^8},$$

that is, one volt is equivalent to 10^8 c.g.s. units of electromotive force.

The following table gives the numerics of the various practical units in terms of c.g.s. units :—

Name of Quantity.	Practical Unit.	Equivalent in c.g.s. Units.
Resistance	Ohm	10^9
Electromotive Force	Volt	10^8
Current Strength	Ampere	10^{-1}
Quantity of Electricity	Coulomb	10^{-1}
Electrostatic Capacity	{ Farad	10^{-9}
	Microfarad	10^{-15}

We have seen above (p. 162) that if Q, Q' be the numerics of two quantities the dimensional formula of $\frac{Q'}{Q}$ is $\frac{[Q']}{[Q]}$, and this of course applies to the expressions of the same qua^r

tity in two different systems of units. Thus if q denotes the numeric of a quantity of electricity in electrostatic units, and Q the numeric of the same quantity in electromagnetic units, the same fundamental units being employed in both cases, the dimensional formula of $\frac{q}{Q}$ is $\frac{[q]}{[Q]}$. But from the table (p. 182) we have $[q] = [M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}]$ and $[Q] = [M^{\frac{1}{2}}L^{\frac{1}{2}}]$. The dimensional formula of $\frac{q}{Q}$ is thus the same as that of velocity, that is to say $\frac{q}{Q}$ may be properly expressed as a *certain definite* velocity, the numerical expression of which depends on the fundamental units of length and time employed. In other words the number of *electrostatic* units of electricity which is equivalent to one *electromagnetic* unit is numerically equal to this velocity.

The same velocity is derivable from the ratios of the numerics of any of the other electrical or magnetic quantities in the two systems of units. For instance, if e be the numeric of an electromotive force in electrostatic units, and E the numeric of the same electromotive force in electromagnetic units, we have for the dimensional formula of $\frac{e}{E}$ the value $\frac{[e]}{[E]} = \frac{[M^{\frac{1}{2}}L^{\frac{1}{2}}]}{[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}]} = [L^{-1}T]$. The ratio $\frac{e}{E}$ has thus the dimensional formula of the reciprocal of a velocity, and therefore $\frac{E}{e}$, or, which is the same, the number of electromagnetic units equivalent to one electrostatic unit of electromotive force, is properly expressed as a certain definite velocity. It is easy to see that this velocity is identical with the former. For if q and Q be the numeric of a certain quantity of electricity, then, since e and E

denote the same electromotive force, the work $e\,q$ must be numerically equal to the work $E\,Q$. We get therefore $\dfrac{E}{e} = \dfrac{q}{Q}$, that is, the two velocities are the same.

Denoting this velocity by v,[1] we get for the various quantities the following relations. The numerator of the ratio on the left of each equation denotes the numeric of the quantity in electrostatic units, the denominator the numeric of the same quantity in electromagnetic units.

A given Quantity of Electricity $\qquad \dfrac{q}{Q} = v$

,, Current $\qquad \dfrac{c}{C} = v$.

,, Electromotive Force $\qquad \dfrac{e}{E} = \dfrac{1}{v}$

,, Electrostatic Capacity $\qquad \dfrac{c_p}{C_p} = v^2$

,, Resistance $\qquad \dfrac{r}{R} = \dfrac{1}{v^2}$

Therefore if q and Q, e and E, or the numerics of any other given quantity, be determined in the two systems of units, the value of v can be at once obtained. Experiments of this kind have been made by Maxwell, Sir W. Thomson, Weber, Ayrton and Perry, J. J. Thomson,[2] and others, with the result that $v = 3 \times 10^{10}$ centimetres per second approximately, or very nearly the velocity of light in air as deduced from experiments made by the methods of Foucault and Fizeau. According to Maxwell's Electromagnetic Theory of Light (*Electricity and Magnetism*, vol. ii.,

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[1] For illustrations of the physical meaning of v see Maxwell's *Electricity and Magnetism*, vol. ii., chap. xix.

[2] J. J. Thomson's result communicated to the Royal Society 1883. is $v = 29\cdot63$ Rayleigh ohms or $2\cdot963 \times 10^{10}$ centimetres per second, the Rayleigh ohm being assumed correct. This result we have reason to believe is the most accurate which has yet been obtained.

chap. xx.) this relation should hold, and thus the theory
is so far confirmed.

Full information regarding experiments for the deter-
mination of v will be found in Maxwell's *Electricity and
Magnetism*, vol. ii., chap. xix. To this work also we
refer the reader who wishes to obtain a complete account
of the mathematical theory of electrical and magnetic
measurements. He may consult also portions of Sir
William Thomson's *Reprint of Papers on Electrostatics
and Magnetism;* Mascart and Joubert's *Leçons sur l'Élec-
tricité et le Magnétisme;*[1] and Prof. Chrystal's articles on
Electricity·and Magnetism,[2] which contain an admirable
digest of the whole mathematical theory together with
much valuable information as to experimental results.

[1] An English translation of this work by Prof. Atkinson has lately been
published.
[2] *Encyclopædia Britannica*, New Edition.

NOTE A. "ON THE DETERMINATION OF H."

THE method given on p. 17 for the determination of the correction for the non-uniform magnetization of the deflecting magnet, gives of course only a first approximation to the true correction, but under the condition that the length of the bar is sufficiently small in comparison with the distance r, say from $\frac{1}{4}$ to $\frac{1}{10}$ of r, and on the supposition that the magnet is reversed at the position on either side of the needle (Fig. 1), it is generally sufficient. The distances should be taken so that $r' = 1\cdot3\ r$ approximately.

In general no such point as a "centre of gravity of magnetic polarity," that is, a point at which the whole of the free magnetism in the actual distribution in each half of the magnet may be supposed to be concentrated so as to have the same action on the needle as actual distribution has, in strictness exists ; but just as we are in the habit of reckoning the centre of gravity of a body which does not strictly speaking possess one as at its centre of inertia, so we may regard the centre of inertia of a material system following the same law of distribution as that of the magnetic matter on either half of the magnet as the "centre of gravity" of that magnetic matter. This point or "pole" is accurately the centre of the horizontal parallel magnetic forces acting on the particles of ideal free magnetic matter on either half of the magnet when it is hung horizontally in the earth's field, as in the vibration experiments, and is deflected from the magnetic meridian.

A method of observation and reduction which amounts to determining the distribution of ideal magnetic matter over the surface of the magnet which has the same external magnetic action as the actual distribution, is given in Maxwell's *Electricity and Magnetism*, vol. ii., chap. vii., where also will be found full information as to the determination of corrections necessary in an exact determination of H.

Let two deflections be taken (Fig. 1) by reversing the deflecting magnet at a distance r_1 on the west side of the needle, and similarly two deflections at the same distance on the east side, and let D_1 be the mean of the tangents of these four deflections. Let this process be repeated at a second distance r_2, and let D_2 be the mean tangent for that distance. It is easy to prove that, approximately, $\dfrac{2\,M}{H} = \dfrac{r_1{}^5\,D_1 - r_2{}^5\,D_2}{r_1{}^2 - r_2{}^2}$. It has been shown (Maxwell's *Electricity and Magnetism*, vol. ii., p. 101 ; or Airy's *Magnetism*, p. 68) that if approximately $r_1 = 1\cdot32\ r_2$, the effect of errors in the observed deflections on the value of $\dfrac{M}{H}$ will be a minimum for these distances.

If long thin bars are used in the determination of H, their magnetic distribution could first be determined very accurately by Rowland's method, and the proper corrections applied. On the other hand, short thick bars of hard steel have the advantage of giving greater magnetic moment for a given length, and they can therefore be placed at a greater comparative distance from the needle, so that the correction for the distribution becomes of less importance. So far, then, as the deflection experiments are concerned it is better to use thick short magnets of the hardest steel, and to place them at such

a distance from the needle that the error caused by neglecting l becomes vanishingly small. On the other hand the magnets must be sufficiently long and thin to render it possible to determine with accuracy their moments of inertia, and therefore to reduce correctly the results of the deflection experiments.

When l is small in comparison with r, we may use the approximate formula

$$M = \frac{r^3}{2} H \tan \theta.$$

for the position shown in Fig. 1 ; or

$$M = r^3 H \tan \theta.$$

for the position shown in Fig. 2.

A magnetic survey of horizontal force in the neighbourhood of a place for which H has been determined may very readily be made with one of the magnets used in the deflection experiments, by simply observing its period of vibration at the various places for which a knowledge of H is desired. The magnetic moment m of the magnet being of course known from the previous experiments, H can be found by equation (5) or (7) of Chap. II. above.

By keeping a magnetometer set up with lamp and scale in readiness, the magnetic moments of large magnets can be found with considerable accuracy by placing them in a marked position, at a considerable distance from the needle, and observing the deflection produced. By having a graduated series of distances, for each of which the constant $\left(\dfrac{r^3 H}{2} \right.$ or $r^3 H$ as the case may be) by which $\tan \theta$ must be multiplied to give M has been calculated, the magnetic moments can be very quickly read off.

The magnetic moments of large magnets of hard steel well magnetized can be determined very conveniently with considerable accuracy by hanging them horizontally in the earth's field, and determining the period of a small oscillation about the equilibrium position. They should be hung by a bundle of as few fibres of unspun silk as possible, at least six feet long, so that the effect of torsion may be neglected. The suspension thread should carry a small cradle or double loop of copper wire, on which the magnet may be laid to give it stability, and to allow of its being readily placed in position or removed. Two vertical marks are fixed in the meridian plane containing the suspension thread, and the observer placing his eye in their plane can easily tell very exactly when the magnet is passing through the equilibrium position, and so determine the period. Or a north and south line may be drawn on the floor or table under the magnet, and the instant at which the magnet is parallel to this line observed by the experimenter by standing opposite one end of the magnet and looking from above. The value of M is given in terms of H by equation (3), Chap. II. above.

Care must of course be taken to avoid undue disturbance from currents of air, and to prevent the magnet, when being deflected from the meridian, from acquiring any pendulum swing under the action of gravity. The deflection from the meridian should be made with another magnet brought, with its length along the east and west line through the centre of the suspended magnet, near enough to produce the requisite deflection, and then withdrawn in the same manner.

NOTE B. "ON POTENTIAL AND LINES OF FORCE."

The electric field due to any distribution of electricity is the whole surrounding space through which the action of the electrified system extends; and the electric force at any point is the force of attraction or repulsion which a unit of positive electricity would experience if placed at the point without disturbing the electric distribution of the system. We may imagine the electric field to extend to an infinite distance from the system, and for infinitely distant points the electric force is of course zero.

The potential at a point in an electric field is the work done by or against electric forces in carrying a unit of positive electricity from the point in question to an infinite distance, the electric distribution being supposed to remain unchanged.

If a quantity q of positive electricity be contentrated at a point O, and P be a point at a distance r from O, the potential at P due to q is $\frac{q}{r}$. For the force of repulsion on a unit of positive electricity at a distance x from O is $\frac{q}{x^2}$, and the work done by this force in increasing the distance by a small length dx is $\frac{q}{x^2} dx$; and summing for all the elements of the path we get for the total work the value $\frac{q}{r}$.

The difference of potentials between P and another point P' at a distance r' from O, or the work done in carrying a unit of positive electricity from P' to P, is therefore $\frac{q}{r} - \frac{q}{r'}$. It is important to remark that this value depends only on the distances of the points from O, and not at all on the path pursued between P' and P.

Further, if we have a number of quantities q_1, q_2, q_3, &c., of electricity at distances r_1, r_2, r_3, &c., from P, the potential at P is $\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + $ &c., as it is usually written $\Sigma \frac{q}{r}$, where Σ denotes summation of a series of terms each of the form $\frac{q}{r}$. Hence the difference of potentials between P and P' is in this case $\Sigma \frac{q}{r} - \Sigma \frac{q}{r'}$. If the distribution of electricity considered be continuous throughout any space, or over a surface, the summation becomes integration through the space, or over the surface.

We remark that the value of $\Sigma \frac{q}{r} - \Sigma \frac{q}{r'}$ depends only on the *positions* of the points P, P'.

A surface every point of which has the same potential is called an *equipotential surface*, or sometimes a *level surface*. Such a surface can evidently be drawn for every point of the electric field.

Any equipotential surface may be taken as the surface of zero potential, and the potential at any point is then simply the difference of potentials between the point and that surface.

Positive electricity tends to move in virtue of electric force from places of higher to places of lower potential, and negative electricity in the reverse direction. If we have a constant flow from one of two conductors joined by a wire to the other, the quantity of electricity transferred per unit of time is proportional to the difference of potentials between the conductors. A difference of potentials between two points or two equipotentials surfaces, when thus considered with reference to flow of electricity from one to the other, is generally called the *electromotive force* between the two points or surfaces. It is to be remarked that this quantity (see Chap. XII.) is not of the nature of a force. The two words "electromotive force" are to be taken *together* simply as a name for the difference of potentials considered in the connection stated.

A curve drawn in the electric field so that the direction of the tangent at any point is the direction of the resultant force at that point is called a *line of force*. Such a line can be drawn through every point in the field.

Since no work is done in carrying a unit of positive electricity from one point of an equipotential surface to an adjacent point, we see that lines of force meet equipotential surfaces at right angles, or in other words that the direction of the resultant electric force at any point of such a surface is normal to the surface.

Consider a small element (of area ds) of an equipotential surface, and imagine lines of force to be drawn through every point in its periphery, so as to form a tubular surface. Such a surface is called a *tube of force*. Let R be the resultant electric force at the element ds, and let $R\,ds$ be taken as unity, the tube is then a *unit tube*.

Imagine any finite area of the surface to be divided into elements, such that the tube for each is a unit tube, and n be their number so that $\Sigma R\,ds = n$, then n is the number of lines of force which cross the area, or, as it is usually put, "the number of lines of force" crossing the area is n. It is to be remembered that what is called a line of force when "number of lines of force" is spoken of is a unit tube.

Let any closed surface be drawn in an electric field due to a distribution of electricity part of which may be without, and part within, the surface, and let N be the normal component in the outward direction of the resultant electric force at a point P on a small element ds of the surface. $N\,ds$ has been called by Maxwell the electric induction at P. Let $\Sigma N\,ds$ be the sum of the products, $N\,ds$, for every element of the surface, then $\Sigma N\,ds$ is what Maxwell has called the *surface integral of electric induction* over the surface. By an important theorem, due to Green, $\Sigma N\,ds$, is equal to $4\pi Q$, where Q is the *algebraic* sum of the quantities of electricity within the surface.

By considering the projection of ds on an equipotential surface at the element, we see that $N\,ds$ is really the same thing as the number of lines of force which cross ds, and as N is to be taken as positive where the lines leave the surface, and negative where they enter it, we see that the excess of the number of lines of force which leave the surface over the number which enter it is equal to 4π times the algebraic sum of the electricity within the surface.

All the definitions and theorems given in this note, as to electric field, electric potential, lines of force, and electric induction, apply also *mutatis mutandis* to magnetism, ideal blue magnetic matter being substituted for

positive electricity, and red magnetic matter for negative electricity, and a unit blue magnetic pole for that of positive electricity.

Imagine an infinitely thin bar magnet, whether straight or curved, magnetized at every point in the direction of the axis with an intensity inversely proportional to the cross-sectional area. The equivalent distribution of ideal magnetic matter is a certain quantity (equal numerically, if the bar is straight, to the magnetic moment divided by the length of the bar) of red matter spread over one end face of the bar, and an equal quantity of blue matter on the other end face. Such a distribution of magnetism is said to be *solenoidal*, and the bar is called a *magnetic solenoid*. The constant product of the intensity of magnetization into the cross-sectional area is called the *magnetic strength* of the solenoid. We may suppose the magnet to be built up of an infinite number of infinitely short pieces, each having the same magnetic strength, placed end to end with unlike faces in contact. The opposite magnetisms on each pair of faces in contact neutralize one another, and we are left with simply the two unbalanced end distributions of the complete bar.

If the ends of the solenoid coincide, that is, if its axis forms a closed curve, the magnet has no external action, and the surface density of the equivalent distribution of ideal magnetic matter is everywhere zero.

When the direction and intensity of magnetization (p. 176) are the same for every point of a magnet, the magnet is said to be *uniformly* magnetized. Every uniformly magnetized body may evidently be considered as made up of solenoids all of the same magnetic strength, and either closed or terminated by the surface of the body.

Now suppose a uniformly magnetized straight bar magnet to be cut across without disturbance to the magnetic distribution into two parts, and these separated in the direction of their length to a very short distance. On the two sides of the crevasse thus formed there will be a uniform distribution of magnetism, exactly similar to that on each end face of the magnet, red on the wall which belongs to the blue pole, and blue on the opposite wall. Then the force on a unit blue pole at any point *in* the crevasse due to the magnetic distribution on the walls would be (p. 177, *note*) $4 \pi \sigma'$, where σ' is the surface density of the distribution of magnetic matter, and is from the blue surface towards the red, that is, *inwards* across a closed surface drawn through the crevasse so as to inclose the half of the magnet to which the blue pole belongs, and the total number of lines of force, or rather of magnetic induction, which cross the surface in that direction is $4 \pi \phi$, where ϕ is the total quantity of magnetic matter on each face. Now since there is just as much red magnetic matter inside the closed surface as there is of blue, the total number of lines crossing the surface is zero, and therefore there must be exactly $4 \pi \phi$ passing *outwards* across the closed surface from the blue magnetism on the end face of the magnet.

The distribution of magnetism in actual magnets is in general not solenoidal, but any magnet whatever can be built up of infinitely thin magnetic solenoids of different lengths, so as to give by their ends the surface distribution equivalent to the actual magnet. Hence if we suppose the bar divided by a crevasse across its neutral region, all the solenoids will be divided, and we can show just as before that the total number of lines of force passing outwards across a closed surface inclosing one part of the magnet is equal to $4 \pi \phi$, where ϕ is the total quantity of magnetic matter in that part of the actual magnet.

This is the theorem we have employed in p. 164. We may here remark that the method there described of determining magnetic distribution is in strictness perfectly accurate only when the magnet is of very small lateral dimensions, and the ring of wire is very thin and fits closely to the magnet. The error is however small in most cases, and the method is greatly superior to the methods generally followed, such as vibrating a small needle

O

at different points near the magnet along its length, or applying a small piece of soft iron at different points of the magnet, and measuring its attraction.

The first of these methods fails to give directly the distribution, because the field at the needle is not due alone to the opposite portion of the magnet, and because of the varying effects of induction at different points which must power-fully affect the needle in a manner which it is impossible to allow for. The second method fails partly because of the fact that the induced magnetization of a piece of soft iron is not in general simply proportional to the magnetizing force, for in fact the iron has different magnetic susceptibilities for different forces and conditions of magnetization. Moreover the soft iron if brought into contact with the magnet alters the magnetic distribution at the point, and thus gives rise to a second source of error.

When the ring of wire is placed at the neutral region of the magnet, and pulled completely off the magnet, we get a very accurate determination of the total quantity of matter in the part swept over, for it is plain that the ring must cut through nearly all the lines of force passing out from that part of the magnet, and this, with a very slight correction for the part of the imaginary closed surface which cuts through the magnet, is equal to $4 \pi \phi$.

If, instead of a solenoidal magnet of finite lateral dimensions and of cylindric shape with ends at right angles to the axis, we take an exactly similar conducting surface, and suppose an electric current of uniform strength per unit of length of the cylinder to flow round it, so that the direction of flow is everywhere at right angles to the generating lines of the surface, we obtain an electromagnetic solenoid, which for all external points is equivalent to an exactly similar magnetic solenoid having at each end a quantity of ideal magnetic matter, numerically equal to $A \gamma$ where A is the cross-sectional area of the cylinder, and γ is the strength of the current per unit of length of the cylinder. This follows from Ampère's theorem (p. 24) of the magnetic equivalence of a small magnet of moment $A C$, and a small closed circuit of area A, in which a current of strength C is flowing. For suppose the cylinder cut across into thin slices, each of thickness $d x$. The current flowing round each element would be $\gamma d x$, and the moment of the equivalent magnet $A \gamma d x$. Now suppose n of these elements placed end to end, they would give a magnet of moment $A \gamma n d x$. But $n d x$ is the length of the corresponding portion of the solenoid, and we have thus for the magnetic moment of the whole electromagnetic solenoid the value $A \gamma l$, and dividing by l we get $A \gamma$ for the total ideal magnetic matter on one end.

Cutting this electromagnetic solenoid into two by a narrow crevasse, and placing a unit pole at any point within, we get as before $4 \pi \dfrac{\phi}{A}$, that is, in this case $4 \pi \gamma$, for the force on the pole due to the magnetic distribution on the two walls; and if the solenoid be long in comparison with its diameter, this will be very exactly the total force at any point within the solenoid over a considerable distance on each side of the centre. Hence the intensity of the magnetic field within the solenoid is $4 \pi \gamma$, the lines of force are parallel to the axis, and a blue pole within is urged towards the blue end of the solenoid. And the total number of lines of force in the cross-section is $4 \pi A \gamma$.

Such a solenoid can be made very approximately by lapping a cylindrical surface round with thin wire so as to form a helix, and sending a current through the wire. If N be the number of spires per unit of length, and C the strength of the current in each, we have $\gamma = N C$, and for the field-intensity at points well within the helix if it be long the value $4 \pi N C$. The total number of lines of force is as before $4 \pi A N C$, where A is the proper mean area of cross-section, which depends on the thickness of wire used and the number of layers. It is this result multiplied by n, the number of turns in the induction coil, that we have used in p. 263.

TABLES.

TABLE I.

CROSS-SECTION OF ROUND WIRES, WITH RESISTANCE, CONDUCTIVITY, AND WEIGHT OF HARD-DRAWN PURE COPPER WIRES, ACCORDING TO THE NEW STANDARD WIRE GAUGE LEGALISED BY ORDER IN COUNCIL, AUGUST 23, 1883.

Temperature 15° Cent.

Descriptive No.	Diameter		Area of Cross-section		Resistance		Conductivity		Weight. (Density=8·95)	
	Ins.	Cms.	Sq. ins.	Sq. cms	Ohms per Yard.	Ohms per Metre.	Yards per Ohm.	Metres per Ohm.	Lbs. per Yard.	Grms. per Metre.
0000000	·500	1·270	·1963	1·267	·000126	·000137	7965	7283	2·285	1134
000000	·464	1·179	·1690	1·091	·000146	·000159	6859	6272	1·970	976·3
00000	·432	1·097	·1466	·946	·000168	·000184	5946	5437	1·706	846·3
0000	·400	1·016	·1257	·811	·000196	·000215	5096	4661	1·463	725·6
000	·372	·945	·1087	·701	·000227	·000248	4409	4031	1·265	627·6
00	·348	·884	·0951	·614	·000259	·000283	3858	3528	1·107	549·6
0	·324	·823	·0824	·532	·000299	·000327	3346	3058	·960	476·1
1	·300	·762	·0707	·456	·000349	·000381	2867	2622	·823	408·1
2	·276	·701	·0598	·386	·000412	·000451	2427	2219	·696	345·4
3	·252	·640	·0499	·322	·000494	·000541	2023	1850	·581	288·0
4	·232	·589	·0423	·273	·000583	·000638	1715	1568	·492	244·1
5	·212	·538	·0353	·228	·000698	·000764	1435	1309	·411	203·8
6	·192	·488	·0290	·187	·000851	·000931	1174	1074	·337	166·8
7	·176	·447	·0243	·157	·00101	·00111	987	902	·283	140·5
8	·160	·406	·0201	·130	·00123	·00134	815	740	·234	116·1
9	·144	·366	·0163	·105	·00151	·00166	661	604	·190	94·0
10	·128	·325	·0129	·0830	·00192	·00210	522	477	·150	74·3
11	·116	·295	·0105	·0682	·00233	·00255	429	392	·123	61·0
12	·104	.264	·00849	·0548	·00290	·00317	345	315	·0989	49·0
13	·092	·234	·00665	·0429	·00371	·00406	270	247	·0774	38·4
14	·080	·203	·00503	·0324	·00490	·00536	204	186	·0585	29·0
15	.072	·183	·00407	·0263	·00606	·00662	165	151	·0474	23·5
16	·064	·163	·00322	·0208	·00760	·00838	131	119	·0374	18·6
17	·056	·142	·00246	·0159	·0100	·0109	99·9	91·4	·0287	14·2
18	·048	·122	·00181	·0117	·0136	·0149	73·4	67·1	·0211	10·4
19	·040	·102	·00126	·00811	·0196	·0215	51·0	46·6	·0146	7·26
20	·036	·0914	·00102	·00657	·0242	·0265	41·3	37·8	·0118	5·88
21	·032	·0813	·000804	·00519	·0307	·0335	32·6	29·8	·00936	4·64
22	·028	·0711	·000616	·00397	·0400	·0438	25·0	22·8	·00717	3·56
23	·024	·0610	·000452	·00292	·0545	·0596	18·3	16·8	·00526	2·61
24	·022	·0559	·000380	·00245	·0649	·0709	15·4	14·1	·00443	2·19
25	·020	·0508	·000314	·00203	·0786	·0858	12·7	11·7	·00366	1·80

TABLE I. (continued.)

CROSS-SECTION OF ROUND WIRES, WITH RESISTANCE, CONDUCTIVITY, AND WEIGHT OF HARD-DRAWN PURE COPPER WIRES, ACCORDING TO THE NEW STANDARD WIRE GAUGE LEGALISED BY ORDER IN COUNCIL, AUGUST 23, 1883.

Temperature 15° Cent.

Descriptive No.	Diameter.		Area of Cross-section.		Resistance.		Conductivity.		Weight. (Density = 8·95)	
	Ins.	Cms.	Sq. ins.	Sq. cms.	Ohms per Yard.	Ohms per Metre.	Yards per Ohm.	Metres per Ohm.	Lbs. per Yard.	Grms. per Metre.
26	·018	·0457	·000254	·00164	·0969	·106	10·3	9·44	·00296	1·47
27	·0164	·0417	·000211	·00136	·117	·128	8·56	7·84	·00246	1·22
28	·0148	·0376	·000172	·00111	·143	·157	6·99	6·38	·00200	·893
29	·0136	·0345	·000145	·000937	·170	·185	5·89	5·39	·00169	·839
30	·0124	·0315	·000121	·000779	·204	·223	4·81	4·48	·00141	·697
31	·0116	·0295	·000106	·000682	·233	·255	4·29	3·92	·00123	·610
32	·0108	·0274	·0000916	·000591	·269	·294	3·71	3·40	·00107	·529
33	·0100	·0254	·0000785	·000507	·314	·343	3·19	2·91	·000914	·453
34	·0092	·0234	·0000665	·000429	·371	·406	2·70	2·47	·000774	·384
35	·0084	·0213	·0000554	·000358	·445	·486	2·25	2·06	·000645	·320
36	·0076	·0193	·0000454	·000293	·544	·594	1·84	1·68	·000548	·262
37	·0068	·0173	·0000363	·000234	·679	·742	1·47	1·35	·000423	·210
38	·0060	·0152	·0000283	·000182	·872	·954	1·15	1·03	·000329	·163
39	·0052	·0132	·0000212	·000137	1·16	1·27	·861	·787	·000247	·123
40	·0048	·0122	·0000181	·000117	1·33	1·49	·751	·672	·000211	·104
41	·0044	·0112	·0000152	·0000981	1·62	1·77	·617	·564	·000177	·0878
42	·0040	·0102	·0000126	·0000811	1·96	2·15	·510	·466	·000146	·0726
43	·0036	·00914	·0000102	·0000657	2·42	2·65	·413	·378	·000118	·0588
44	·0032	·00813	·00000804	·0000519	3·07	3·35	·326	·298	·0000936	·0464
45	·0028	·00711	·00000616	·0000397	4·00	4·38	·250	·228	·0000717	·0356
46	·0024	·00610	·00000452	·0000292	5·45	5·96	·183	·168	·0000527	·0261
47	·0020	·00508	·00000314	·0000203	7·85	8·58	·127	·117	·0000366	·0181
48	·0016	·00406	·00000201	·0000130	12·3	13·4	·0815	·0746	·0000234	·0116
49	·0012	·00305	·00000113	·00000730	21·8	23·8	·0459	·0420	·0000132	·00653
50	·0010	·00254	·000000785	·00000507	31·4	34·3	·0319	·0291	·00000914	·00453

NOTE.—The resistances and conductivities in Tables I. and II. are according to the B.A. determination of the ohm, and are calculated by taking 1.642×10^{-6} B.A. units as the resistance at 0° C., between the ends of a hard-drawn copper wire 1 cm. long and 1 sq. cm. in cross-section. This agrees with Matthiessen and Hockin's result (*B. A. Rep.*, 1864, and *Phil. Mag.*, vol. xxix., 1865) of ·1469 B.A. unit as the resistance at 0° C. of a wire one metre long weighing one gramme, if the density, 8·95, of cast specimens of their copper be taken as approximately the density of the wires experimented on, which was not determined. To reduce the numbers to accord with Lord Rayleigh's determination of the ohm subtract 1·3 per cent. from the resistances and add 1·3 per cent. to the conductivities.

TABLE II.

CROSS-SECTION OF ROUND WIRES, WITH RESISTANCE, CON-
DUCTIVITY, AND WEIGHT OF PURE COPPER WIRES,
ACCORDING TO THE BIRMINGHAM WIRE GAUGE. (See
Note to Table I.) Temperature 15° Cent.

B.W.G.	Diameter.		Area of cross-section.		Resistance.		Conductivity.		Weight. (density = 8·95)	
	Ins.	Cms.	Sq. ins.	Sq. cms.	Ohms per Yard.	Ohms per Metre.	Yards per Ohm.	Metres per Ohm.	lbs. per Yard.	Grms. per Metre.
0000	·454	1·153	·162	1·0444	·000152	·000167	6566	6004	1·884	934·7
000	·425	1·079	·142	·915	·000174	·000190	5754	5262	1·651	819·1
00	·380	·965	·113	·732	·000217	·000238	4601	4206	1·320	654·8
0	·340	·864	·0908	·586	·000272	·000297	3693	3377	1·056	524·2
1	·300	·762	·0707	·456	·000349	·000382	2867	2622	·822	408·1
2	·284	·721	·0633	·409	·000389	·000425	2570	2350	·737	365·8
3	·259	·658	·0527	·340	·000468	·000512	2138	1954	·613	304·2
4	·238	·605	·0445	·287	·000554	·000606	1805	1650	·518	256·9
5	·220	·559	·0380	·245	·000649	·000709	1543	1411	·442	219·5
6	·203	·516	·0324	·209	·000762	·000833	1313	1200	·377	186·9
7	·180	·457	·0254	·164	·000969	·00106	1032	993	·296	146·9
8	·165	·419	·0214	·138	·00115	·00126	867	793	·249	123·5
9	·148	·376	·0172	·111	·00143	·00157	698	638	·200	99·3
10	·134	·340	·0141	·0910	·00175	·00191	572	523	·164	81·4
11	·120	·305	·0113	·0730	·00218	·00238	458	419	·132	65·5
12	·109	·277	·00933	·0602	·00264	·00289	378	346	·109	53·9
13	·095	·241	·00709	·0457	·00348	·00380	288	263	·0825	40·9
14	·083	·211	·00541	·0349	·00456	·00498	219	201	·0630	31·2
15	·072	·183	·00407	·0263	·00606	·00662	165	151	·0474	23·5
16	·065	·165	·00331	·0214	·00749	·00813	135	123	·0386	19·2
17	·058	·147	·00264	·0170	·00933	·0162	107	97·7	·0307	15·3
18	·049	·124	·00189	·0122	·0131	·0143	76·5	69·9	·0220	10·9
19	·042	·107	·00139	·00894	·0178	·0196	56·2	51·4	·0161	8·00
20	·035	·0889	·000962	·00621	·0256	·0280	39·0	35·7	·0112	5·56
21	·032	·0813	·000804	·00519	·0307	·0335	32·6	29·8	·00936	4·64
22	·028	·0711	·000616	·00397	·0400	·0438	25·0	22·8	·00716	3·55
23	·025	·0635	·000491	·00317	·0502	·0549	20·0	18·2	·00571	2·83
24	·022	·0559	·000380	·00245	·0649	·0709	15·4	14·1	·00442	2·19
25	·020	·0508	·000314	·00203	·0786	·0858	12·7	11·6	·00367	1·82
26	·018	·0457	·000254	·00164	·0969	·106	10·1	9·43	·00296	1·47
27	·016	·0406	·000201	·00130	·123	·134	8·16	7·46	·00234	1·16
28	·014	·0356	·000154	·000993	·160	·175	6·24	5·71	·00179	·889
29	·013	·0330	·000133	·000856	·186	·203	5·38	4·92	·00154	·766
30	·012	·0305	·000113	·000732	·218	·238	4·59	4·19	·00132	·653
31	·010	·0254	·0000785	·000507	·314	·343	2·91	2·92	·000915	·454
32	·009	·0229	·0000636	·000410	·388	·424	2·58	2·36	·000746	·367
33	·008	·0203	·0000503	·000324	·491	·536	2·04	1·86	·000585	·290
34	·007	·0178	·0000385	·000248	·641	·701	1·56	1·43	·000442	·220
35	·005	·0127	·0000196	·000127	1·26	1·37	·798	0·728	·000229	·113
36	·004	·0102	·0000126	·0000811	1·96	2·15	·510	0·466	·000146	·072

TABLE III.

CONDUCTIVITIES OF PURE METALS AT $t°$ C.*

Conductivity at o° C. = 1.

Metal	Conductivity at $t°$ C.
Silver	$1 - 0038278t + 000009848t^2$
Copper	$1 - 0038701t + 0000090009t^2$
Gold	$1 - 0036745t + 000008443t^2$
Zinc	$1 - 0037047t + 000008274t^2$
Cadmium	$1 - 0036871t + 000007575t^2$
Tin	$1 - 0036029t + 0000061361t^2$
Lead	$1 - 0038756t + 0000091461t^2$
Arsenic	$1 - 0038996t + 000008879t^2$
Antimony	$1 - 0030826t + 0000103641t^2$
Bismuth	$1 - 0035216t + 000005728t^2$
Iron	$1 - 0051184t + 0000129161t^2$

* From the results of Matthiessen's experiments; and to be used only for temperatures between o°C. and 100° C. The formulas, excluding that for iron, agree closely, and give the mean formula $1 - 0037647t + 00000834ot^2$.

TABLE IV.

CONDUCTIVITY AND RESISTANCE OF PURE COPPER AT TEMPERATURES FROM 0° C. TO 40° C.

Calculated by the formula for the Conductivity of Copper in Table II.

Temp.	Conductivity.	Resistance.	Temp.	Conductivity.	Resistance.
0 °	1·0000	1·0000	21	0·9227	1·0838
1	0·9961	1·00388	22	0·9192	1·0879
2	0·9923	1·00776	23	0·9158	1·0920
3	0·9885	1·0116	24	0·9123	1·0961
4	0·9847	1·0156	25	0·9089	1·1003
5	0·9809	1·0195	26	0·9054	1·1044
6	0·9771	1·0234	27	0·9020	1·1085
7	0·9734	1·0274	28	0·8987	1·1127
8	0·9696	1·0313	29	0·8953	1·1169
9	0·9559	1·0353	30	0·8920	1·1211
10	0·9622	1·0393	31	0·8887	1·1253
11	0·9585	1·0433	32	0·8854	1·1295
12	0·9549	1·0473	33	0·8821	1·1337
13	0·9512	1·0513	34	0·8788	1·1379
14	0·9476	1·0553	35	0·8756	1·1421
15	0·9440	1·0593	36	0·8723	1·1464
16	0·9404	1·0634	37	0·8691	1·1506
17	0·9368	1·0675	38	0·8659	1·1548
18	0·9333	1·0715	39	0·8628	1·1591
19	0·9297	1·0756	40	0·8596	1·1633
20	0·9262	1·0797			

TABLE V.

SPECIFIC RESISTANCES IN B.A. UNITS OF WIRES OF DIFFERENT METALS AND ALLOYS.[1]

Substance.	Resistance at 0° C. of wire one cm. long one sq. cm. in section.	Resistance at 0° C. of a wire one foot long weighing one grain.	Resistance at 0° C. of a wire one metre long weighing one gramme.	Resistance at 0° C. of a wire one foot long, 1−1000th in. in diam.	Resistance at 0° C. of a wire one metre long one millimetre in diam.	Percentage increase of resistance for 1° C. increase of temperature at 20° C.
Silver, annealed	1·521 × 10⁻⁶	0·2214	0·1544	9·151	0·01937	0·377
Silver, hard drawn	1·652 ,,	0·2415	0·1680	9 936	0·02103	...
Copper, annealed	1·616 ,,	0·2064	0·1440	9·718	0·02057	0·388
Copper, hard drawn	1·652 ,,	0·2106	0·1469	9·940	0·02104	...
Gold, annealed	2·081 ,,	0·5849	0·4080	12·52	0·02650	0·365
Gold, hard drawn	2·118 ,,	0·5950	0·4150	12 74	0·02697	...
Aluminium annealed	2·945 ,,	0·1085	0·0757	17·72	0·03751	...
Zinc, pressed	5·689 ,,	0·5831	0·4067	34·22	0·07244	0·365
Platinum, annealed	9·158 ,,	2·810	1·96	55·09	0·1166	...
Iron, annealed	9·825 ,,	1·097	0·7654	59·10	0·1251	...
Nickel, annealed	12·60 ,,	1·535	1·071	75·78	0·1604	...
Tin, pressed	13·36 ,,	1·396	0·9738	80 36	0·1701	0·365
Lead, pressed	19·84 ,,	3·236	2·257	119·39	0·2527	0·387
Antimony, pressed	35·90 ,,	3·456	2·411	216·0	0·4571	0·389
Bismuth, pressed	132·7 ,,	18·64	13·03	798·0	1·689	0·354
Mercury, liquid (see Note)	96·19 ,,	18·72	13·06	578·6	1·2247	0·072
Platinum - Silver Alloy,[2] hard or annealed	24·66 ,,	4·243	2·959	148·35	0·314	0·031
German Silver Alloy, hard or annealed	21·17 ,,	2·652	1·850	127·32	0·2695	0·044
Gold-Silver Alloy,[3] hard or annealed	10·99 ,,	2·391	1·668	66·10	0·1399	0·065

[1] Given by Professor Jenkin as expressing the results of Matthiessen's experiments. The numbers underlined are the results of direct observation.
[2] Two parts platinum, one part silver, by weight.
[3] Two parts gold, one part silver, by weight.

Note.—According to a very careful determination of the specific resistance of mercury made by Lord Rayleigh and Mrs. Sidgwick (*Phil. Trans.*, Part I., 1883), the value given in this table is too high. Their final result is 95·412 × 10⁻⁶ B.A. unit as the resistance at 0° C. of a column of pure mercury one cm. long and one sq. cm. in section. A column of pure mercury therefore one sq. millimetre in section, which at 0° C. has a resistance of one ohm, has, according to the B.A. determination of the ohm, a length of 104·81 cms., and, according to Lord Rayleigh and Mrs. Sidgwick's determination, 106·21 cms.

The value given in Col. I. for hard-drawn copper has evidently been calculated from the corresponding observed result in Col. III., by using the density 8·89 for copper, and is therefore higher than that in which Tables I. and II. are founded. (See *Note* to Table I.)

TABLE VI.

THE APPROXIMATE INTENSITY OF THE HORIZONTAL
COMPONENT OF THE EARTH'S MAGNETISM AT
SEVERAL PLACES IS GIVEN BELOW, BUT IT MUST BE
BORNE IN MIND THAT THE VALUE OF THIS
ELEMENT MAY BE GREATLY ALTERED BY IRON
OR MAGNETS NEAR THE PLACE OF OBSERVATION

Place	Value	Place	Value
Cambridge	·181	Newcastle	·167
London	·181	Dublin	·167
Birmingham	·176	Carlisle	·166
Nottingham	·175	Edinburgh	·162
Stafford	·175	Glasgow	·161
Sheffield	·173	Dundee	·161
Manchester	·172	Aberdeen	·158
Liverpool	·171	Inverness	·156
York	·171		

TABLE VII.

(1) ELECTRO-CHEMICAL EQUIVALENTS.

One Coulomb of Electricity liberates $3 \cdot 31 \times 10 - 4$ of a gramme of Copper.

| ,, | ,, | ,, | ,, | $3 \cdot 21 \times 10 - 4$ | ,, | Zinc. |
| ,, | ,, | ,, | decomposes $9 \cdot 45 \times 10 - 5$ | | ,, | Water. |

(2) THERMAL EQUIVALENTS.

One Gramme-Water-Degree Centigrade is equivalent to $4 \cdot 2 \times 10^7$ Ergs.*

,,	,,	,,	,,	,,	$4 \cdot 2$ *Joules.*
One Pound-Water-Degree Centigrade				,,	$1 \cdot 91 \times 10^{10}$ Ergs.
,,	,,	,,	,,	,,	1390 Foot-Pounds.
,,	,,	,,	Fahrenheit	,,	$1 \cdot 06 \times 10^{10}$ Ergs.
,,	,,	,,	,,	,,	772 Foot-Pounds.

*That is, $4 \cdot 2 \times 10^7$ (42 million) ergs spent in friction would just generate heat sufficient to raise one gramme of water from 0° C. to 1° C.

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THE END.